Proceedings
of the
Seventeenth National Congress
of the
Association for Mathematics Education
of South Africa (AMESA)

"Mathematics in a Globalised World"

11-15 July 2011

University of the Witwatersrand

Johannesburg

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Volume 1

Association for Mathematics Education of South Africa (AMESA)
P. O. Box 54, Wits, 2050, Johannesburg

17th Annual AMESA National Congress, 11 to 15 July 2011; Johannesburg, Gauteng. Volume 1

First published: July 2011

Published AMESA

ISBN: 978-0-620-47378-1

Foreward and acknowledgements

As Editors of the paper submissions for the AMESA 2011 Congress, we open this foreward with a thanks to the Mathematics Education community for continuing to support AMESA through your submission of papers featuring a variety of mathematical topics, educational phases ranging from primary mathematics to undergraduate mathematics learning and mathematical and pedagogical learning within teacher education, as well as a significant number of papers focused on theoretical and methodological issues. International submissions also continue to feature – which suggests that AMESA's profile continues to progress beyond South African borders.

The plenary papers also deal with a range of issues facing mathematics education at this point, in South Africa and globally, and detail some of the opportunities and challenges inherent in working in the field. Marlene Sasman outlines data collected at national and provincial levels on performance at Matric level, and details some of the ways in which this information might be used to understand learner misconceptions and guide interventions. Karin Brodie discusses the principles underlying the teacher development model that is being used within the Data Informed Practice Improvement Project. Gugu Moche shares insights on the historical development of calculus, with a view to understanding how this historical development can illuminate the difficulties faced by undergraduate students on this topic. Stephen Sproule, now located in Tennessee, provides examples of an eclectic range of careers open to students with mathematical skills, and shares some of the ways in which mathematical understandings and tools figure within the problems that need to be solved within these roles.

The 'Hot Topics' plenary panel discussions are focused in this Congress on topical issues within mathematics education. The imminent changes in mathematics curriculum suggested within the new Curriculum and Assessment Policy Statements (CAPS) across all phases was one such issue which we felt was important for AMESA to engage with – the panel session led by Michael de Villiers focuses on the geometry in the curriculum across all phases within the CAPS documents. Following on from Marlene Sasman's plenary, and linked to Umalusi's work on analysis of South African Matric Mathematics papers over a number of years, in relation to international benchmarks, Lynn Bowie leads a panel discussion on how to draw upon this expertise more widely to improve the learning of mathematics. The final panel discussion is a conversation with four AMESA past presidents, focused on their views on AMESA's role in relation to supporting research and development work in mathematics education. Papers from each of the past presidents (Moses Mogamberry, Aarnout Brombacher, Mamokgethi Setati and Ray Duba) are included in this proceedings.

We offer particular thanks to the authors of our plenary, panel and long and short papers, for your prompt submissions – this has allowed us to get all accepted papers into these proceedings this year. Additionally, and importantly, we also thank all of you who have supported this work through your reviewing of papers, and in some instances, your input into supporting authors with revisions. This kind of work is central to AMESA's mandate to support development in the field of mathematics education. All papers were peer reviewed by two individuals in the mathematics/ mathematics education communities – a list of all reviewers is provided.

We have been supported hugely in the work of reviewing, collating and putting together the proceedings for this Congress by Jessica Sherman and Rencia Lourens, who coordinated the large number of workshop and How I Teach submissions. There too, they took on board the remit to work developmentally with authors and have supported the editing of submissions.

We have learnt a lot from the process of being involved in the putting together the Academic Programme for AMESA 2011. We hope that you find the Conference enjoyable and useful to your ongoing work in the field.

Hamsa Venkat & Tony Essien

June 2011

Many thanks to the following who volunteered their time in reviewing the papers sent to them in a constructive and helpful spirit:

Alwyn Olivier
Bruce Tobias
Caroline Long
Lynn Bowie
Craig Pournara
D Sampson
Devika Naidoo
Erna Lampen
Gerrit Stols
Helen Sidiropoulos
Hennie Boshoff
Humphrey Atebe
Iben Christiansen
Ingrid Sapire
Jacques du Plessis
Jayaluxmi Naido

Jill Adler
Karin Brodie
Kim Ramatlapana
Lovemore Nyaumwe
Lyn Webb
Marc North
Marc Schafer
Margot Berger
Mellony Graven
Judah Makonye
Michael de Villiers
Mike Mhlolo
Hamsa Venkat
Patricia Nalube
Percy Sepeng
Sarah Bansilal

Belinda Huntley
Shaheeda Jaffer
Sharon Mcauliffe
Sibawu Siyepu
Surgeon Xolo
Vera Frith
V. G. Govender
Werner Olivier
Willy Mwakapenda
Zain Davis
Aarnout Brombacher
Stephen Sproule
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PLENARY PAPERS

INSIGHTS FROM NSC MATHEMATICS EXAMINATIONS

Marlene Sasman

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There is a growing concern in South Africa about the state of mathematics teaching in South Africa and the mathematical skills and abilities students have when leaving school. In this paper I delve into the responses and performances of students in the National Senior Certificate examinations and use the analysis to present some of the findings regarding the obstacles to improvement in school mathematics..

INTRODUCTION

The National Senior Certificate is the gateway to Higher education and thus all spheres of South Africa's society regards the National Senior Certificate results as the main indicators of the quality of the education system. There is thus an understandable concern about what these results are saying about the current state of education.

A variety of stakeholders in society exert demands on school mathematics. These include parents, learners, teachers, mathematics educators employers, professional mathematicians, tertiary institutions, professional, cultural and political lobbies and organisations. Stakeholders influence mathematics classrooms to a greater or lesser extent. There is always a potential for conflict between the rationale for mathematical education embedded in mathematical education as envisioned by mathematics educators, and demands exerted by other stakeholders.

Mathematics is seen as an important indicator of the system since it is undoubtedly true that a strong mathematical background is necessary for many career and job opportunities in an increasingly technological society. Most tertiary institutions require Matric Mathematics as a prerequisite for many fields of study. Sometimes the actual mathematical demands of these fields of study do not justify these prerequisites.

An important purpose of mathematics is the establishment of proper connections between mathematics as a discipline and the application of mathematics in real-world contexts. Mathematics provides learners with the means to analyse, understand and describe their world and to deepen their understanding while adding to the ability to solve real-world problems. As such mathematics is important for the economy and nation building.

Historically 30% to 40% of secondary schools in the country simply did not offer any mathematics beyond Grade 9, and thus, the full potential for students with possible passes in mathematics was not realised. With the introduction of the National Curriculum Statement all Further Education and Training (FET) phase students are required to take some form of mathematics. Approximately half of the FET candidate population is currently taking Mathematics whilst the rest take Mathematical Literacy.

This special role of mathematics has seen it continuously singled out in discussions on curriculum reforms and analyses of examination results to determine the state of education. Concerns about the quality of mathematics passes have become a thorny issue since the quality of performance is often linked to the quality of teaching and is the subject of much debate. Data in South Africa in terms of links between teachers' level of content knowledge and their ability to teach the curriculum competently remains small scale, and thus claims and assumptions are often uninformed.

South Africa has adopted an internationally renowned definition which refers to Mathematics as "… human activity…" (DOE curriculum statements, 2002; 2003). The pedigree of this definition can be traced from what is currently known as the Freudenthal Institute of Mathematics in the Netherlands where the notion of 'mathematics as human activity …' was coined (Freudenthal, 1968).

A critical analysis of this definition has the following fundamental implications for teaching and learning of mathematics:

- mathematics must be connected to reality.
- stay close to children and be relevant to society.
- provision of "guided" opportunity to "re-invent" mathematics by doing it.
- emphasis should be on horizontal and vertical 'mathematization'.

Teachers' knowledge and understanding of these approaches as well as properly orchestrating them, are essential in preparing students for further understanding of mathematical concepts. This implies that teachers' content knowledge as well as appropriate methodology are essential prerequisites for preparing students appropriately and to ensure that students acquire the fundamental skills of mathematical proficiency.

In order for students to achieve high quality educational outcomes, we need to have evidence about the performance of our students' performance. Literature on evidence-based interventions views national and international tests as important sources of authentic information that can be used to structure educational interventions, particularly content and instructional related interventions. An analysis of the National Senior Certificate (NSC) examination results is used as a major source of evidence.

RESULTS

Year	No. Wrote Math	No. Wrote Math HG		No. Passed Math HG	
1995	200,000	60,000	30.0%	29,000	14.5%
1997	231,000	68,000	29.4%	20,000	8.7%
1999	281,000	50,000	17.8%	20,000	7.1%
2001	264,000	35,000	13.3%	20,000	7.6%
2003	258,000	36,000	14.0%	23,000	8.9%
2005	303,000	44,000	14.5%	26,000	8.6%
2007	348,000	46,000	13.2%	25,000	7.2%

Table 1: Analysis: 1995 – 2007 (Only alternate years shown to show trend)

The above table clearly indicates that Mathematics Higher Grade (HG) was fast becoming an elitist subject. According to some research 70% of the passes came from 11% of schools.

We come from a history where Mathematics Higher Grade was seen as being elitist. The difference between Mathematics Higher Grade and Mathematics Standard Grade was of such a nature that it was much easier to teach and learn the latter and many learners were left with no option but to take Mathematics Standard Grade. The dwindling number of Mathematics Higher Grade candidates presented disastrous implications for the economy of the country which needs candidates with mathematics backgrounds for professions such as engineering, health and economic services, etc.

Year	Number who wrote Mathematics	Number Passed	% Passed	Variance in Entries
2008	300 829	141 953	47. 2 %	-
2009	301 654	138 353	45 . 9 %	+ 825
2010	263 034	124 749	47 . 4 %	– 38620

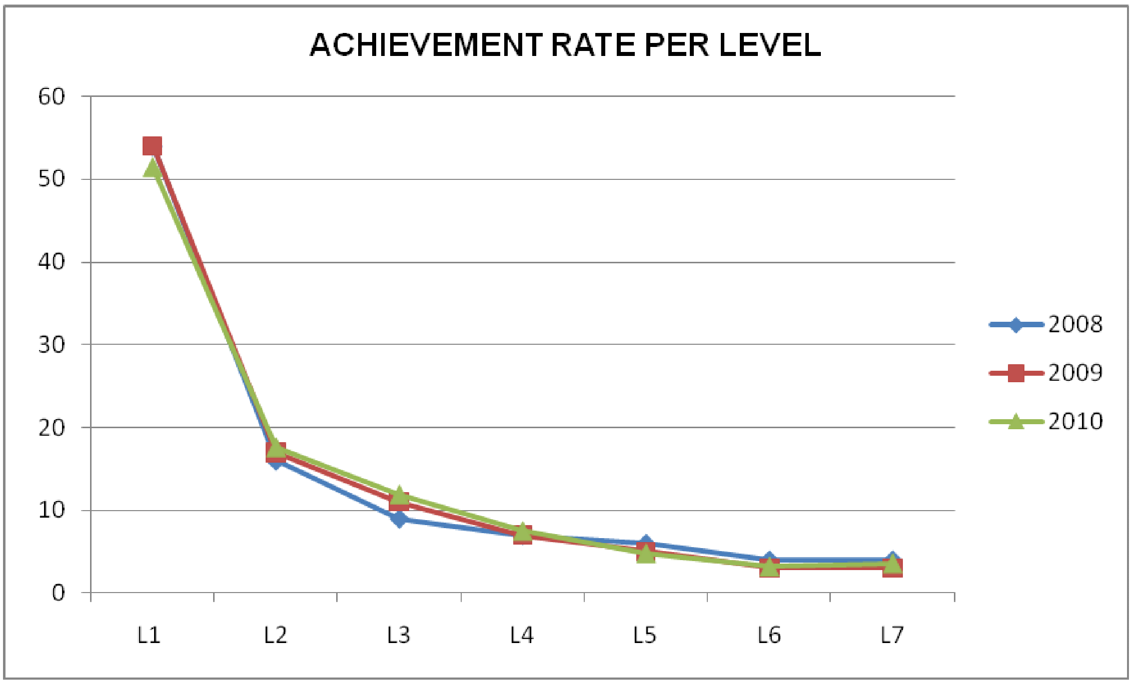
Table 2: Analysis: 2008-2010

The 2008 Mathematics examination was preceded by sets of exemplar papers and was perceived by many to be ‘easy’. The 2008 examination was the first examination based on the revised National Curriculum Statement and maybe examiners were over-cautious in setting an examination that was accessible to learners across the vast range of ability since there was no longer differentiation between different levels of Mathematics. Perhaps they did not want to deter students from mathematics which may lead to an unequal balance between Mathematics and Mathematical Literacy which would be detrimental to the needs of the country. The pass rate however does not confirm this perception that the paper was ‘easy’. Also most passes were at lower levels. What the country needs is more candidates passing Mathematics at levels 6 and 7.

The 2009 examination was much debated and perceived as ‘extremely difficult’. This could be the reason for the significant drop in students registering for mathematics. Once again the statistics did not support the view that the examination was ‘extremely difficult’ since the pass rate differed by only 1.3 %. The 2010 examination was perceived as cognitively comparable with 2009 yet the pass rate improved slightly.

Table 3: Quality of Pass rate: 2008 -2010 (in %)

	L1 0-29%	L2 30-39%	L3 40-49%	L4 50-59%	L5 60-69%	L6 70-79%	L7 80-100%
2008	54	16	9	7	6	4	4
2009	54	17	11	7	5	3	3
2010	51.4	17.6	11.9	7.5	4.8	3.2	3.6



From the analysis of the data above it is clear that the quality of the passes indicates a consistent poor trend and this is a serious cause for concern.

2008: 70% of the candidates obtained between 0% and 39%

2009: 71% of the candidates obtained between 0% and 39%

2010: 69% of the candidates obtained between 0% and 39%

The results reveal performance in Mathematics which is consistently poor with the majority of candidates passing at the lower end. A bell curve, or normal distribution would have most candidates achieving percentages or levels in the middle range: 40% to 50% or levels 3 and 4.

The analyses below reveal that poor patterns of performance may be because most candidates are only able to handle 'stimulus-response' questions and questions at the lower cognitive levels but illustrate little conceptual understanding.

FACTORS IMPACTING ON PERFORMANCE IN MATHEMATICS

Analysis of responses in the NSC examination

The tables below present data on the provincial average percentage obtained in each question in the 2009 – 2010 NSC Mathematics papers from the Western Cape Education Department. The concomitant topic per question has been included.

Paper 1

Paper 1 - 2009		
	Ave %	Topic
Question 1	50	Algebra
Question 2	30	Patterns, Sequences and Series
Question 3	44	
Question 4	31	
Question 5	14	
Question 6	18	Functions
Question 7	23	
Question 8	14	
Question 9	30	Finance
Question 10	53.7	Calculus
Question 11	35	
Question 12	15	
Question 13	30.6	Linear Prog.
TOTAL	**29.4**	

Paper 1 - 2010		
	Ave %	Topic
Question 1	56.4	Algebra
Question 2	44.1	Patterns, Sequences and Series
Question 3	29.5	
Question 4	44.6	Functions
Question 5	43.6	
Question 6	24	
Question 7	22.9	Finance
Question 8	36.6	Calculus
Question 9	20.1	
Question 10	15	
Question 11	30	Linear Prog.
TOTAL	**34.1**	

- The responses to the questions on algebra revealed poor understanding of the basics and foundational competencies taught in earlier grades e.g. algebraic manipulation, properties of numbers, factorisation, solution of equations and inequalities. In particular many candidates lacked an understanding of the subtle differences and theoretical understanding between solving quadratic equations versus solving quadratic inequalities. They over-generalised and treated the solution of quadratic inequalities in a purely algorithmic way as they would equations. Many have no understanding of the meaning of variable. This understanding should evolve over the grades from Foundation phase to the Further and Education and training phase:

$4 + \square = 7$	- Foundation phase: the unknown as placeholder
$x - 3 = 12$	- Intermediate Senior phase: the unknown as placeholder
$x + y = 15$	- Senior/FET phase: the unknown as variable
$6\cos^2 x - 5\cos x = 4$	- FET phase: the unknown as variable and also recognising structure as $6p^2 - 5p = 4$

Kieran (1989) emphasises that an important aspect of this difficulty is students' difficulty to recognise and use structure. Structure includes the "surface" structure (e.g. that the expression 3(x + 2) means that the value of x is added to 2 and the result is then multiplied by 3 and the "systemic" structure (the equivalent forms of an expression according to the properties of operations, e.g. that 3(x + 2) can be expressed as (x + 2) × 3 or as 3x + 6).

- The Patterns, Sequences and Series questions indicated that candidates struggled to answer questions asked indirectly and questions embedded in words. They were unable to convert flexibly from words to symbols. Many illustrated only an operational understanding of infinite sequences being able only to write down the formula and find the sum but unable to solve problems. A lack of understanding of sigma notation was clearly evident. In both examinations candidates were asked to determine the general term for a given sequence. Most candidates did not realise the necessity of validating their generalisations in terms of the given data.
- Many candidates did not have a clear understanding of the characteristics of various families of functions and are unable to sketch graphs. The link between patterns and functions should be made explicit in the teaching such that a linear/arithmetic pattern is understood as an equivalent representation of the line graph. The curriculum requires candidates to be able to convert flexibly between words, tables, graphs and symbols. Candidates also lack an understanding of the behaviour of functions. The notation embedded in functions and the transformations of functions were poorly understood in 2008 and 2009. 2010 has seen an improvement in these aspects.

- The performance in the questions based on finance needs much attention. Here again the link between geometric sequences and the derivation of appropriate formulae for annuities should be made explicit. Many learners and educators are under the impression that it is just the matter of choosing a formula to solve these problems and are unable to solve problems where they have to set up an equation using appropriate formulae. The lack of an understanding of timelines which could assist in solving problems is evident.
- In every examination candidates revealed deficient understanding in the application of differential calculus. Solution attempts suggest that understanding of calculus is based on practicing algorithmic procedures devoid of understanding. Many are able to find the derivative from first principles and by means of rules in an algorithmic way (Q 10 : 2009 and Q 8 : 2010) because of sufficient practice but do not understand what they have calculated. In these questions their errors in handling notation and the multiplicity of rules indicate a lack of understanding. Their inability in the application of calculus indicates a very poor conceptual grasp and is widespread even among the candidates who perform well.
- Although Linear Programming has been removed from the Curriculum Policy and Assessment Statement (CAPS, DOE, 2011) the problems associated with this topic are prevalent in other aspects of the curriculum. A salient feature of this topic is the ability to set up the constraints symbolically from words and then to represent them graphically. The well-drilled candidate may be able to do this if given the constraint but is then unable to deal with interpretive questions – again an indication of well-rehearsed algorithmic procedures with little understanding. Poor literacy levels often prevent candidates from accessing the mathematics embedded n these problems.

A comment by a post matric student: "I hated algebra because I did not understand it - but I was good at it" - sums up the fact that students could get through the school system with only well-rehearsed algorithmic skills but little conceptual understanding. This is perhaps linked to the excessive use of calculators which prevents students from understanding the structure of the algorithms for the addition, subtraction, multiplication and division of fractions, for example. Understanding algebra depends on understanding structure. The FET NCS overall places strong emphasis on developing conceptual understanding: ***'The emphasis is on the objective of solving problems and not on the mastery of isolated skills.'***

Paper 2

Paper 2 - 2009		
	Ave %	Topic
Question 1	54	Statistics (Data-handling)
Question 2	62	
Question 3	34	
Question 4	46	Analytical Geometry
Question 5	40	
Question 6	38	Transformation Geometry
Question 7	22	
Question 8	36	Trigonometry
Question 9	36	
Question 10	10.7	
Question 11	15	
Question 12	14	
TOTAL	**34.1**	

Paper 2 - 2010		
	Ave %	Topic
Question 1	64.7	Statistics (Data-handling)
Question 2	51	
Question 3	50.8	
Question 4	53.4	
Question 5	41	Analytical Geometry
Question 6	34.2	
Question 7	40.2	Transformation Geometry
Question 8	18.6	
Question 9	44	Trigonometry
Question 10	27.4	
Question 11	15.3	
Question 12	20.8	
TOTAL	**34.1**	

- In 2008 the Data-handling questions were at the end of the examination. This appeared to hamper optimal performance in these questions which were in many cases very accessible. This underlined the fact that if accessible questions come after many cognitively demanding questions it disadvantages students.

  Some learners especially second language learners, found it difficult to express themselves. They could not identify which statistical calculation should be utilized in making appropriate comparisons.

  More emphasis should be placed on the interpretation and analysis of statistical data. Candidates must be exposed more often to questions where they have to effectively communicate conclusions drawn from an analysis of data. In many instances candidates appeared not to understand the terminology associated with this section.

- In classical mathematics, analytical geometry, also known as coordinate geometry, or Cartesian geometry, is the study of geometry (*points*, straight *lines*, and *circles* being among the basic objects) using a coordinate system and the principles of algebra and analysis. However the responses revealed how compartmentalised our teaching is. For example many learners do not know that to find the point of intersection of two graphs, say a straight line and a circle, they have to solve two equations simultaneously although this is introduced in earlier grades as well as in algebra. Separating algebra from graphs is like removing milk from tea. They are inextricably bound. A graph is a representation of an equation; the study of equations is algebra.

- Candidates struggled with finding the general rule for a combination of transformations performed in sequence and accompanying notation in this regard.

  In determining the area of a triangle on a Cartesian plane candidates did not use the "dissection" method i.e. drawing a rectangle around the triangle, finding its area and subtracting the area of the extra right angled triangles formed in the process from the rectangle although is part of the explorative work on space and shape done in earlier grades. Candidates mainly calculated sides and sometimes did not use the correct sides in the area formula for a triangle: A = $\frac{1}{2}$base x height. It is as if candidates cling to formulae in the higher grades since this is how many perceive mathematics. It is also suspected that some candidates actually measured the sides of ΔA′B′C′ with a ruler although the instructions at the beginning of the paper stated that diagrams were not necessarily drawn to scale. The teaching process should expose students to the integration of algebra, transformation and analytical geometry. The concept of rotation through a general angle depends on the understanding of compound angles – which in the previous curriculum was higher grade work. This is an indication that this work is cognitively more demanding and that the average candidate will struggle with this question, unless very well prepared.

- Trigonometry was the most poorly answered section in Paper 2.There are still instances where some candidates showed no knowledge of the basics of trigonometry. It is a matter of concern that a learner can spend 3 years in the FET band and ostensibly have learnt nothing in trigonometry. Candidates struggled to identify the relevant quadrant, struggled with reduction formulae, compound and double angles, and co-functions. It was also disturbing to note that quite a few learners obtained a negative value for the radius.

  Poorly developed spatial perception impeded the solving of 2D and 3D problems

  In grade 10 the NCS states: "Understands that the similarity of triangles is basic for the trigonometric functions sin α , cos α and tan α, and is able to define and use the functions."

  Similarity has its basis in the understanding the concept of ratio and proportion which has its origins in earlier grades. It is thus clear that unless these important concepts are competently taught in the earlier grades we will continue to have consistent poor performance in trigonometry.

Commenting on the mark adjustments made to the Matric Mathematics results, Umalusi in 2010 stated in their public report (2011):

> With Mathematics becoming compulsory, approximately half of the candidate population (296 000) took Mathematics, hence the lower performance than in the previous SC higher grade. The need for admission to higher education urges the choice of mathematics as subject as well. This

together with other systemic issues results in a performance in Maths which is consistently poor. In 2008 the adjusted mean resulted in an increase from 22,69% to 28,26%, in 2009 the adjustment brought about an increase from 20,79% to 27,3%. The raw means of 2008 and 2009 confirmed the qualitative reports which indicated that the 2009 papers were of higher cognitive demand and more compliant with the SAG than in 2008. The raw mean of 23,66% in 2010 is similar to the raw mean of 2008. The 2010 papers were compliant with the SAG requirements.The scaled adjustment in 2010 resulted in a mean of 28,6% which is aligned with the mean of previous years. The pairs analysis reveals that the Mathematics question papers were significantly more difficult than the other subjects.

The report confirms that there has been an upward adjustment every year since 2008.

Other perceptions from candidates' responses

Many of the errors and misconceptions gleaned from learners responses have their origins in a poor understanding of the basics and foundational competencies taught in the earlier grades. This suggests that interventions to improve candidates' performance should focus also on knowledge, concepts and skills learnt earlier and not just on the final year of the FET phase.

Many schools are not affording the new approaches to certain content topics the appropriate teaching and assessment time. The reason for this could be that many educators are not confident about some content in the NCS – also some educators appear to be struggling with the depth to which new topics as well as some of the older topics must be taught and thus often spend too much time on teaching certain sections which they know well and teaching the less familiar topics at a very superficial level. **This results in educators struggling to teach the entire curriculum.**

It also appears that many educators are struggling to handle the diversity of ability in the class and **concentrate on the lower cognitive levels and often do not expose stronger candidates to the necessary challenge required**. This is evident from the fact that many candidates struggled to do questions that required deeper insight.

From analyses of internal school-based and some provincial examinations it is very clear that many students are dismally unprepared for the challenges of the Grade 12 Mathematics examination. Questions are often lifted from external papers but concentrates on level 1 and 2.

Besides the new topics such as Financial Mathematics many experienced educators have only been exposed to teaching the standard grade curriculum in the past. This could be the reason why candidates lacked the necessary insight to deal with certain questions demanding a deeper conceptual understanding.

Contextual questions in examinations serve as barriers to some learners because of poor literacy levels and often prevent them from identifying the mathematical skills involved. This is unfortunate since an important purpose of Mathematics in the FET Band is *the establishment of proper connections between mathematics as a discipline* *and the application of mathematics in real-world contexts.*

CONCLUSION

Our performances in international mathematics and numeracy assessments such as TIMSS have also indicated consistently poor performance from South Africa. There are many other factors that impact on performance. In the analysis of the TIMSS tests it was indicated that the one factor that consistently affected performance amongst all participating countries were socio-economic factors.

We also have a shortage of appropriately trained mathematics educators with schools then using under qualified teachers to teach Mathematics, especially the earlier grades. This often results in an impoverished curriculum being delivered with poor foundational competencies resulting in learners being ill prepared for mathematics in the higher grades. Learners often pass from grade to grade without passing Mathematics. A large number of learners should perhaps have been doing Mathematical Literacy but parents, for obvious reasons, are reluctant to let their children follow this route. The role of Mathematics versus Mathematical Literacy or another level of Mathematics should be investigated.

The new draft Curriculum and Assessment Policy Statement (CAPS) for Mathematics has increased the content and concomitant cognitive levels for the vast majority of learners. This raises the potential danger of reverting to a previous scenario where the majority of learners in disadvantaged communities are only offered Mathematical Literacy.

The continuous dismal performance has also affected teachers' morale and belief in their learners' abilities. We need urgent intervention and training which will address:

- Teaching strategies and methodology
- Content knowledge and understanding
- Planning to ensure curriculum-completion
- Motivation and interest for both students and teachers

Research has shown that our curriculum is not more demanding than that in other countries. We need to have confidence in our educators and if they lack sufficient content knowledge, urgent steps need to be put in place to address the problem.

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IT'S AMAZING WHAT YOU CAN DO WITH MATHEMATICS

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While the number of careers using mathematics continues to blossom only a few of them have been shared in this paper. Six categories of mathematically rich careers have been identified. These are data mining, notional analysis, engineering, mathematical modelling, operations research and computer algorithm writing. Importantly, the mathematics teacher is identified as the central figure in firstly opening access to these careers for his or her learners and secondly inspiring learners to develop a passion for a career in mathematics.

INTRODUCTION

In our modern world, mathematics has become ingrained in the functioning of almost every sector of the economy and the burgeoning range of careers open to mathematicians accommodates the interests of most learners who also show an interest in mathematics. The careers described in this paper are only a small sample of the amazing opportunities that are developing for mathematicians. With the world changing so quickly, the opportunities that will exist for your learners can't even be imagined today. The influence of high speed computers and the growth of new fields on the career opportunities of mathematicians cannot be underestimated. Increasingly careers in mathematics require the mathematician to not only solve problems but also to formulate the questions. This means that while learners should have a strong mathematical knowledge they should also be developing an ability to apply that knowledge and making judgments about when to use what mathematical ideas.

In this paper I have organised the mathematics careers into six broad categories. While there is much overlap in the mathematical tools used in these categories, the purposes to which the mathematics is directed vary significantly. The six categories are data mining, notional analysis, engineering, mathematical modelling, operations research and computer algorithm writing. A discussion of actuarial careers using probabilistic mathematics has not been included in the paper. I have not described actuarial science as a career simply because it seems to get the most attention already and space for the paper is limited. In each category I discuss a few of the more interesting careers available to our learners.

The first step for learners to become involved in mathematical careers is for them to imagine what is possible and for that to happen you need to have a vision of greatness for your learners. While South Africa is desperate for engineers and scientists, we can do nothing without YOU, the mathematics teacher!

DATA MINING

In a modern, computerised world, data are being generated and stored at a pace never seen before. Statisticians are increasingly being tasked to mine the large volume of data to search for patterns of behaviour and use these to predict future behavior (Ayres, 2007). Search engines such as Yahoo, Bing or Google already keep individual search activity on file for anything from 6 to 18 months. At present no one is interested in what you do as an individual just what you do as part of a collective who all behave with regularity and consistency.

Household electronic devices are increasingly turning digital and being connected to the Internet making your behaviour accessible for data mining. The result is greater convenience and better service to the customer while at the same time allowing data miners to use the data to narrow your experiences and control your choices. One day your web-connected refrigerator will influence the advertisements you see on the Internet, keep your doctor informed about your health, write your shopping list for you, tell you which food stuffs are going off and maybe even suggest guests for your next party.

In America statisticians mine and then analyse the web surfing behaviour of the people who click on advertisements so that they can build a profile of the type of people who may be interested in certain products. People with similar surfing behaviours who visit the website in the future will then face targeted rather than random advertisements. For example, data miners at a company called Tacoda search for patterns of web surfing behaviour in over 20 billion clicks every day (Baker, 2009). They noticed that people who clicked on car rental advertisements also most commonly clicked on obituary listings (Baker, 2009). This made sense because often people with a death in the family have to fly to the funeral and then rent a car in the distant city. What was strange was that the second most common connection to car rentals was people who had also viewed listings of romantic movies. This connection required further investigation. It turns out that people who watch romantic movies like the thought of travelling to romantic destinations. So those searching romantic movies websites will now find car rental advertisements on their favourite sites.

The next great development in the medical field will be applying the mathematical lens to medical evidence to make better diagnoses of patients' ailments and identify potential ailments earlier so that they have a better chance of being cured (Baker, 2009). It won't be long before improved and cheaper medical devices will be available in your home and will transmit data about your health to your computer and then, via the Internet, to your physician. Firstly, your personal data captured while you are healthy will form a health baseline that can then be used to identify the first signs of variation from the baseline hoping to catch potential ailments early on. For example, once your daily routines are established, variations in when you get out of bed, how long it takes for you to answer the phone and other daily behaviours will be tracked and may be used to produce early identification of the onset of diseases like

Alzheimer's.

Secondly, mining the data of millions of people will radically alter the medical field forever. Evidence for the success or failure of medicines will multiply rapidly. Connecting patterns of related conditions or seeking significant variations in medical data will help us stay healthy but will also increasingly intrude into our lives. Data driven medical decision making will dominate the medical profession, sadly reducing the role of the doctor but supposedly improving the medical services we receive. Some biologists are so enthralled with the potential of the gadgets and mathematical models that they have called mathematics, 'the biologist's next microscope only better' (Ayres, 2007).

A research team at the University of Washington decided to fight back against erratic airline pricing (Ayres, 2007). The results of their mathematical endeavours were used to create a website called Farecast.com which predicted whether the price of a flight was likely to increase or decrease in the near future (Ayres, 2007). Farecast has since been be bought by Bing travel. Using custom made software and powerful computing they continuously collect millions of flight prices, searching for patterns in the pricing trends of flights. This data is then analysed and a mathematical model is used to predict the trend of the price of a particular flight. The model is also adjusted as the results of the predictions are tested against actual trends in the price of flights and a basket of simulated purchases performed by the computers specifically to test the models. Customers to bingtravel.com are then simply given an up or down arrow, and a level of confidence in the prediction (figure 1).

Figure 1. The price predictor on the Bing travel website.

The purpose of data mining is not to be able to say "I know what you did last summer", it is more about saying "I know what you're planning next summer." The mathematics involved is high end statistics that can be used to look for patterns in behaviour as well as behavioural deviations from the norm. These are used to build predictive mathematical models. Data miners and quantitative analysts need a strong background in statistics and probability as well as enough mathematics to allow them to build the predictive models. General problem solving abilities are also important so that they can probe connections between data that others might ignore or not be able to explain.

NOTATIONAL ANALYSIS

Notational analysis is an objective way to answer subjective questions related to performance. While it is used in a number of service industries and the arts it has become most associated with the study of performance in sports. The purpose of notational analysis is to keep a tighter track on sport tactics, techniques and performance. At its simplest, researchers can use a grid of the playing field (figure 2) and notate the player actions they want to study onto the grid. Usually a computer with very expensive software is used to annotate patterns of play, player movement and performance onto a schematic which is then associated with the video footage of the match and a wide range of summary data is produced (Hughes & Franks, 2008). Critical coaching questions are then studied with objective data produced by the computer quickly and effectively. The type of information that may be produced includes the direction of attack on goal in a hockey match, which of the forwards are arriving late to the ruck in a rugby match, or what weaknesses are evident in an opponent's tennis shots during a tournament.

The coaches now have objective information with which to coach the players and rectify team strategy as needed during the week's training sessions. For example, figure 2 illustrates the outcome of corners taken by a team over a number of soccer matches. Since the team is only scoring from balls played to the back post, it is clear that the team needs a new strategy for attacking the front post from corners.

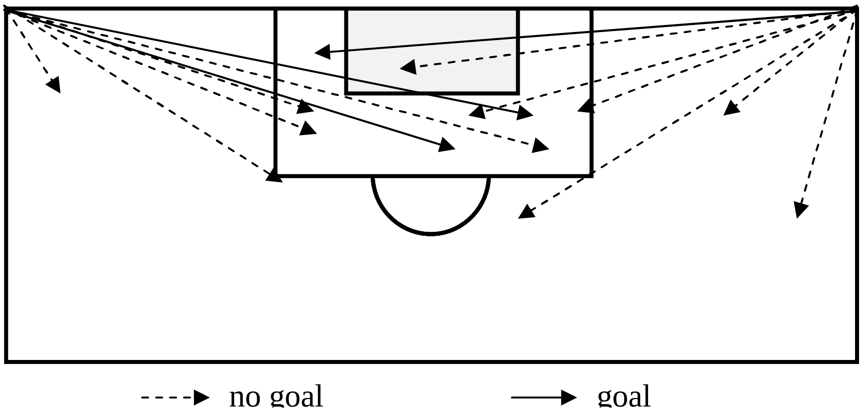


Figure 2. A notational analysis of corners taken in a soccer match.

A few years back the results of notational analysis were given to the coaches a few days after the match and then used to coach players, develop better team strategies or find weaknesses in the strategies of upcoming opponents. With high speed computing, coaches are increasingly able to get feedback during the match. For example, using small GPS transmitters we can now keep track of how far a soccer or hockey player runs during the course of a match. Coaches can then keep an eye on players who have run significantly more than their teammates, improving substitution decisions near the end of a match.

Prior to a soccer match, goalkeepers are given a summary of the penalty taking habits of the opposition players. If a penalty taker has a bad habit of favouring a particular side of the net the modern goalkeeper will know about it. To alleviate this problem, strikers taking penalties often have interesting rituals to try to randomise the side of the net they will shoot at from a penalty (Eastway & Haigh, 2007).

Much of this information is never shared with the viewing public but we are slowly starting to see more of it on our television screens or available on the Internet. In cricket the hawk-eye spread of balls bowled is now available on www.cricinfo.com.

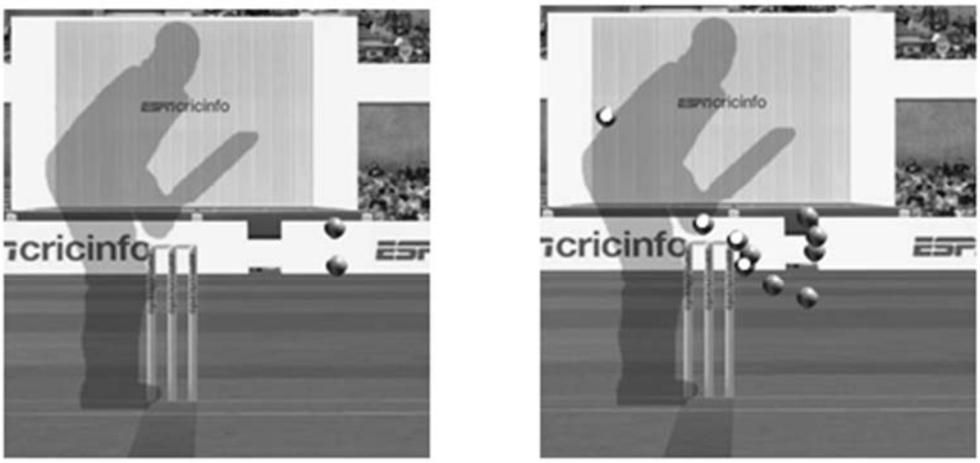


Figure 3. Boundaries by left-handed batsmen (a) South Africa, (b) New Zealand. Balls with a white dot represent onside boundaries, the others represent offside boundaries (ESPNCricinfo, 2010).

A simple analysis of the hawk-eye in figure 3 can be quite informative about South Africa's loss to New Zealand in the recent cricket world cup. Considering that South Africa's four left hand batsmen included Graeme Smith, JP Duminy and Morné Morkel, all generally big hitters, it is very telling that between them they only managed two offside boundaries and no onside boundaries in South Africa's quarterfinal match. As illustrated in the hawk-eye images in figure 3, the New Zealand bowlers managed to avoid bowling into the legside hitting zone of these powerful legside players and dried up their runs.

The strength the mathematician brings to notational analysis includes the ability to ask the right questions, collect reliable data and analyse the data mathematically to produce valid and reliable information to the coach. The mathematics involved is mostly simple statistics, including correlation and significance testing, that is learned in first year university. Recently researchers have tried working with more sophisticated techniques, such as chaos theory and fuzzy logic (Hughes & Franks, 2008).

ENGINEERING

Engineering continues to be one of the dynamic careers favoured by those with a passion for mathematics. In our modern world a degree in engineering no longer means that you will work in traditional engineering positions. The skills of an engineer are sought after by a wide range of industries and businesses, even in the financial sector. Increasingly, creative engineers are using their strong analytic ability to work in design engineering, working on anything from household appliances to the aesthetics of buildings.

Research engineers may use more sophisticated calculus including differential equations and three dimensional multivariable calculus but the working civil engineer mostly uses simple calculus, lots of ratio and proportion, high school algebra and a strong problem solving ability. In fact engineers often have to pose their own mathematical problems to develop a solution to a real world dilemma.

An example of the mathematics required by an engineer is the civil engineer's use of the parabola to design a cut-and-fill road. In mountainous areas engineers often blast through a hill to build a more level road and then use the rubble that has been cut from the hill to fill the valley, again flattening out the roadway (figure 4) and making it more easy to travel.

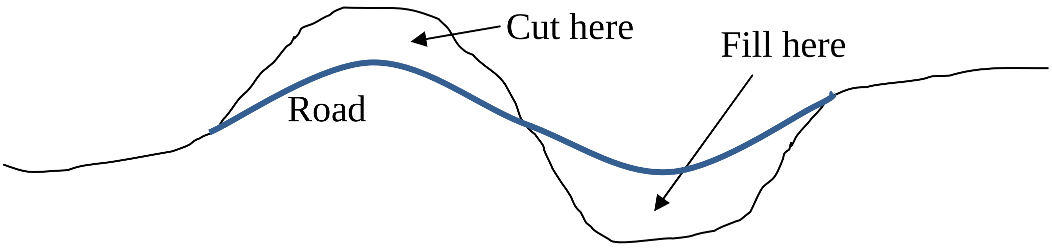
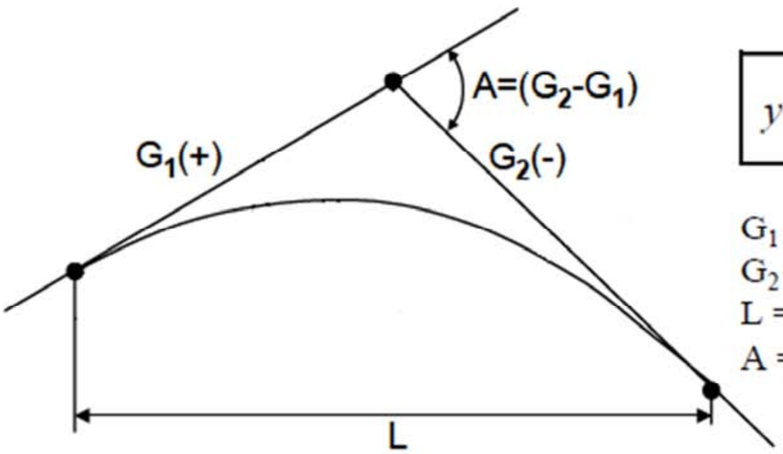


Figure 4. Side view of the cut-and-fill construction of a mountain road.

There are some wonderful examples of cut-and-fill roads on the N3 between Harrismith and Pietermaritzburg, look out for them on your next journey. Fascinatingly, the calculation for designing a more level road requires a knowledge of contour maps, gradient and the parabola. Figure 5 illustrates the parabolic curvature of a leveled road and the formula for the parabola that is used to design the road.



$$y = \frac{G_1 x}{100} + \frac{(G_2 - G_1)x^2}{200L}$$

G_1 = % grade of tangent 1
G_2 = % grade of tangent 2
L = KA (length of vertical curve)
A = (G_2-G_1) (algebraic diff of grades)

Figure 5. The parabolic vertical curvature of a mountain road

MATHEMATICAL MODELLING

Mathematical models are usually built by a mixture of inductive study of data obtained from empirical research and deductive mathematical reasoning used to make decisions about what variables are important. Once the model has been developed it is always tested in simulations against real data or situations to evaluate its effectiveness. Since the purpose of all models is predicting the future a model is said to be only as good as its predictions are accurate. Most of us never encounter the actual models just the predictions. Probably the most common modeled prediction we experience is the weather forecast every evening.

Mathematical models are used extensively to predict changes in the world's stock markets. Some New York brokers rely solely on the decisions of their mathematical models to buy or sell shares, particularly derivative based products. The blame for the huge housing and then banking crunch of 2008 has often been laid at the feet of the quantitative analysts of the big investment houses in cities like New York and London. The quants, as the quantitative analysts are often called, thought their mathematical models could compensate for the blatant flouting of basic economic principles that was happening in investment houses. They were wrong.

Mathematical models are evident in other aspects of the business world such as inventory forecasting in the retail world. The forecasting models do much of their work based on historical sales data from individual stores but also input variables from other sectors of the business. The teams of mathematicians involved must interact with the marketing department, logistics, warehousing and purchasing (Macphee, 2000). While the models that produce sales forecasts are well established, the statistician still uses his or her knowledge when unforeseen circumstances arise or problems need to be solved. The mathematics required is often only early university statistics but more importantly the ability to ask good questions and know what mathematics to use to answer the questions is critical.

Mathematical modelling is also increasingly being used to search for patterns in environmental data so that we can predict natural phenomenon and disasters. The cost, sturdiness and reliability of electronic scientific probes that can communicate wirelessly have allowed mathematicians faster and more immediate access to data so that their models can make more accurate and useful predictions. For example, the world's oceans are now dotted with buoys containing equipment that measures currents, wave properties and weather data and then feeds this data via satellite to central computers. Scientists are then able to map sea currents, temperature, winds, air pressure, wave height and a number of other attributes (Breivik & Allen, 2008).

In the past when a mayday signal, containing the latitude and longitude of a vessel in distress, was received during a storm over the ocean, sea rescue would dispatch helicopters or planes to the location of the mayday signal. The planes would then fly in a grid pattern from that location searching for the vessel. This has changed in the past 5 years. Today when the mayday is received the location of the vessel in distress is fed into a computer which then uses the ocean currents and wind information coming from nearby buoys to model the movement of the vessel. The model uses stochastic processes and produces a probability density function that evolves with time to give a best estimate of the location of the vessel for up to 60 hectares into the future (Breivik & Allen, 2008). This process has dramatically improved the rescue of sea crews saving many lives in the process.

Mathematicians in collaboration with other scientific fields are now also modelling avalanches, tsunamis, earthquakes and volcanic activity, using their models to predict events before they happen so that they can give sufficient warning to people needing to move out of danger. Current efforts are underway to model and predict the volcanic activity near Goma in the DRC.

OPERATIONS RESEARCH

Operations research, or OR as we like to call it, has been described as ‘the science of better’ (Informs, undated). OR is the process of applying (sometimes rather complex) mathematical tools to find optimal solutions that narrow the choices from a wide range of feasible choices to presenting a few ‘best’ solutions so that the people involved in the operation can choose which one works in their context.

Most OR activities start with a problem, sometime a problem that the clients aren’t even able to articulate very well. The operations research team then formulates the problem into a mathematical problem by identifying the necessary variables and choosing appropriate mathematical tools to collect data, analyse the problem and calculate solutions. Computer simulations are often used to test the outcomes of the solutions before they are implemented by the client (Informs, undated). Most OR solutions are tailored made to the situation and people involved. While the number crunching may be done by a machine the operations research team is responsible for identifying the most pertinent variables and mathematizing the question so that optimal solutions can be presented to the client.

In industry, the modern warehouse design and operation is fully run by operations research solutions. Operations researchers track the inflows and outflows of the products to optimise the size of the warehouse, the warehouse layout, the packing distribution and the retrieval mechanisms (Askin & Standridge, 1993). Managing the logistics of the warehouse may also involve using quieter times to move items that are now selling reduced volumes to the back of the warehouse so that new and faster selling items can be placed near the entrance and so reduce retrieval times for loading the distribution trucks. The mathematics used includes such topics as linear programming and 3D analytic geometry. Applying mathematics to managing a warehouse can save workers time and employers up to 40% on the costs of running the warehouse

South Africa’s Air Traffic and Navigation Service (ATNS) is responsible for about

10% of the world's airspace and is highly regarded as one of the best airspace managers in the world (Metron Aviation, 2010). With only three international airports already running near capacity, the expanded flight schedule during the Soccer World Cup placed intense pressure on the systems at both our airports and the ATNS, where the number of aircraft in the sky over South Africa nearly doubled. Our ATNS called in the operations researchers and implemented an air traffic flow management system that enabled decision making supported by mathematical solutions (Metron Aviation, 2010). During the Soccer World Cup, the ATNS safely managed a total of 108 120 aircraft arrivals and departures at 20 airports across the country. In addition implementing the mathematical transport solutions saved ATNS over R16 million (Metron Aviation, 2010).

Operations researchers usually have a degree in applied mathematics or actuarial science and are known for their broad mathematics backgrounds. They usually have expertise in an eclectic mix of statistics, probability, numerical methods such as game theory and queuing theory, linear programming, pure mathematics, computer algorithm writing and the like.

COMPUTER ALGORITHM WRITING

All software packages rely on mathematical algorithms to function but some use algorithms as an active part of the functionality of the software. These range from complex CAD packages used in architecture and engineering to computer games. Amazing innovations in software in the last few years have opened career possibilities for mathematicians to use algorithms to produce graphics for animation and computer games. The graphics of computer games as well as computer created animations are highly reliant on the mathematics of physics. The discerning viewer wants to see a realistic representation of paint splatter, a car tumble or a monster stomp through a sinister forest. Mathematicians are employed to write the algorithms that ensure animated sequences behave and look like the real thing but with dramatic effect. Careers in animation and graphics live right on the edge between mathematics and art, and give those with an artistic flair but mathematical bent a wonderful career outlet. In addition to animation in movies the same mathematical role is played in the animated graphics of computer games.

In essence algorithms are mathematical models requiring a knowledge of applied mathematics tools such as mechanics, differential equations, matrices and eigenvectors. For example, a strong knowledge of fluid dynamics is needed to mathematically model the animation of fluid, fire, air, smoke, snow or even the movement of animal fur (Joyce, 2002). Knowledge of mechanics is needed for programing movement, objects falling or flying and ensuring that a puddle forms correctly from rain coming in through an open window.

In the movie, The Curious Case of Benjamin Button, the head of Benjamin Button was a digital head for nearly the first 60 minutes of the movie (Ulbrich, 2009). The movie is about a man, Benjamin, who was born old and grew physically younger as he aged. The animation of the face broke new ground, made possible by more powerful computers, mapping 100 000 polygons rather than the traditional 100 polygons (Ulbrich, 2009). As illustrated in figure 6, cameras mapped Brad Pitt's face and produced a smooth digital human head. The digital head was then aged using a mathematical model to predict what Brad Pitt might look like at 80.

Figure 6. Transforming the face of Brad Pitt into an aged digital face.

The one advantage of working in this field is that the model is evaluated by the visual effect it produces rather than all the subtle predictive powers that are expected of many other mathematical models. While efficiency in programming is always important, with an appealing realistic image the programmer can get away with a little clumsiness in this job. In animation graphics perception is truly reality.

CONCLUSION

The near global drive for productivity and systematic control of business functions means that the Twenty-first Century belongs to the mathematician. Of course the use of mathematics can get a little carried away. For example, the British spent good money researching and developing a mathematical model to calculate the perfect crunchiness of the cooked bacon on the bacon sandwich or butty (Cowell, 2007).

Remember that the only person who can give your learners access to these exciting new careers in mathematics is you, the mathematics teacher. Your teaching is required for them to learn the necessary mathematical knowledge and develop their reasoning, your encouragement for them to persevere and your inspiration for them to grow bright eyes and dream of the possibilities.

A critical component of mathematics careers in the modern world is that they work in tandem with other areas of expertise. Learners who might be good at mathematics but have a real passion for another field should be encouraged to add as much mathematics to their studies as possible. After all mathematics gives you the edge!

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TEACHER LEARNING IN PROFESSIONAL LEARNING COMMUNITIES

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In this paper I share some principles from the Data Informed Practice Improvement Project, which has been developed at Wits University over the past few years. The project develops an innovative model of teacher development, with three main strands: teachers work with data from their classrooms, they use this data to understand and engage with learner errors, and they do this collectively in professional learning communities, with facilitation from members of the project team. I describe each of these strands and present two examples that illuminate how the strands work together to support teacher learning in professional communities.

INTRODUCTION

The current view of mathematics teacher professional development is that professional development programmes that focus on teacher learning in and from practice are more likely to result in lasting changes in teaching practices (Borko, 2004; Kazemi & Hubbard, 2008). Six key characteristics of successful professional development programmes have been identified: they are long-term and developmental; they focus on artifacts of practice such as student thinking, tasks and instructional practices; they use actual classroom data; they encourage design and reflection on the part of teachers; they are job-embedded (school-based) and therefore blur the boundaries between teaching and learning about teaching and they promote the development of professional learning communities. The effectiveness of such professional development programmes is believed to lie in supporting teacher collaboration in order to produce shared understanding, a focus on curriculum and instruction, and being of sufficient duration to ensure progressive gains in knowledge (Little, 1993).

Our project is innovative in South Africa in that it takes all six of the above criteria into account. Two key strands of our project: working with classroom data in professional learning communities, bring together the six characteristics to produce innovative teacher development work in South Africa. In this paper I describe some of the key principles of our project and show how the project activities work to support teacher learning in teacher professional communities.

PROFESSIONAL LEARNING COMMUNITIES

The term ‘professional learning communities’ usually refers to teachers “critically interrogating their practice in ongoing, reflective and collaborative ways” in order to promote and enhance student learning (Stoll & Louis, 2008, p.2). For Katz and Earl (2010, p.28) professional learning communities are "fundamentally about learning – learning for pupils as well as learning for teachers, learning for leaders, and learning for schools" and for Louis and Marks (1998, p.535) they support teachers to "coalesce around a shared vision of what counts for high-quality teaching and learning and begin to take collective responsibility for the students they teach". While these definitions capture the essence of the idea – professionals engaged in ongoing learning for the benefits of their "clients"- they are also relatively broad and allow for a range of teachers, educators and researchers to use the term with different meanings and consequences, some of which are contradictory (Hargreaves, 2008). So, clearer definitions are necessary to indicate exactly how the term is being used.

Stoll and Louis (2008, p.3) review the literature and present a definition which highlights key aspects. The term 'professional learning community' suggests that the focus is not just on individual teachers' learning but (1) on professional learning; (2) within the context of a cohesive group; (3) that focuses on collective knowledge; and (4) occurs within an ethic of interpersonal caring that permeates the life of teachers, students and school leaders. Stoll and Louis emphasize the collective and caring nature of professional learning communities. However, they do not discuss the contents of the learning or methods that members of the professional learning community use to engage in learning. Katz, Earl and Ben Jafaar (2009) identify four key characteristics of successful professional learning communities: (1) they have a challenging focus; (2) they create productive relationships through trust; (3) they collaborate for joint benefit, which requires "moderate professional conflict", although not personal conflict; and (4) they engage in rigorous enquiry. For Katz et al. successful learning communities are those that challenge their members to reconsider taken-for-granted assumptions, because this is the only way that genuine change can happen. This is why professional conflict is to be encouraged, as it promotes rigorous inquiry. In order for professional conflict not to become personal conflict, an ethic of care and trust is necessary. McLaughlin and Talbert (2008) distinguish between teacher communities which maintain traditional practices and the status quo, and those which re-invent and re-invigorate practice.

While collaboration, rigorous inquiry, trust and care are necessary, they are not sufficient for successful professional learning communities. A crucial element is the focus, or content – what the community collaborates to inquire into, or what is being learned. The research suggests that in order to have the greatest effect on student learning, the focus must relate to the instructional core – the relationship between teacher, student and content (City, Elmore, Fiarman, & Teitel, 2009) and involve a problem of practice based on learner needs (Boudett, City, & Murnane, 2008). For example, students might be able to solve simple, single-step problems in mathematics, but cannot go further to solve multi-step problems (learner need). If teachers choose this as their focus, they need to work out why learners struggle with multi-step problems, what in their current teaching has not helped learners to solve multi-step problems (problem of practice), and how they might shift their teaching to help learners. A key question for professional learning communities is how do they establish a “clear, defensible focus” (Katz, et al., 2009, p.23), that is “right, shared and understood” (Katz, et al., 2009, p.47).

THE ROLE OF DATA

A clear and defensible focus or a problem of practice needs to be established on the basis of data. Such data can and should come from a wide range of sources – national or international test results, teachers’ own tests, interviews with learners, learners’ work and classroom observations. The use of data provides a means whereby teachers can identify real learner needs, as opposed to their own intuitions as to what learners need. Data might confirm what teachers know to be the problem, or they might suggest a problem that teachers do not know about.

An important distinction needs to be made here between evidence-based practice and data-informed practice. Evidence-based practice is associated with a movement that argues that only research-based evidence is good enough to inform what teachers need to learn. Such an approach ignores the relevance of school and classroom-level data and teachers’ own knowledge of their learners. It also expects that teachers read research, which we know is not the case. Data-informed professional development suggests that teachers themselves, with some expert guidance, can and should interpret the full range of data that is available to them and that their interpretations will suggest areas of learners’ learning needs and hence teachers’ learning needs.

But the data alone is not enough. Local data has to be brought into contact with current knowledge and research in order for teachers to find ways forward with their problems of practice. Jackson and Temperley (2008) argue for a model where practitioner knowledge - in particular knowledge of subject, learners and the local context - meets public knowledge, which is knowledge from research. These come together to form new knowledge, which is both research-based and locally relevant.

There is general agreement that in order to truly shift practice in ways that support learner improvement, teachers must be willing to challenge their own practice and give up long-held beliefs if these are seen to not be working. This is what makes the work difficult. At the same time there is also general agreement that no single practice or set of practices is necessarily the “right” one in any school. Although there is general support for practices which support learners’ conceptual understanding and reasoning, Hargreaves (2008) makes an important point when he says that conventional wisdom about teaching must be considered in relation to current research and what the data shows – intuition and craft knowledge must come into contact with research knowledge so that both can be interrogated.

THE DATA INFORMED PRACTICE IMPROVEMENT PROJECT (DIPIP)

The Data-informed practice improvement project (DIPIP) is a professional development project that works with teachers to build and sustain professional learning communities in which teachers engage with data from a range of sources and work together to better understand the nature of learners' errors and how they might respond to them. In the first two phases of the project (2008-2010) we piloted a number of activities that help teachers to do this: test analysis, interviewing of learners and curriculum mapping (Brodie, Shalem, Sapire, & Manson, 2010). Based on these activities teachers identify a key concept that seems to be problematic for learners. They then read and discuss research on learners' thinking in those concepts (Molefe, Brodie, Sapire, & Shalem, 2010), design lessons to engage with learners' thinking, and then videotape and reflect on those lessons (Brodie, 2011; Brodie & Shalem, in press). In the current phase of the project, DIPIP Phase 3 (2011-2012) we are working in schools with mathematics departments, using the activities to build the departments' collective engagement with data from their schools, and their design of and reflection on lessons based on their data-analysis. We ran a pilot for Phase 3 in one school in 2010 (Chauraya & Brodie, 2011). In both phases of the project, teachers met/meet weekly. In Phases 1 and 2 they worked across schools at Wits, led by Wits facilitators and in Phase 3 they work in their schools, also led by Wits facilitators, who travel to the schools once a week. The idea in Phase 3 is to hand over leadership of the professional learning communities to teachers in the schools so that they become ongoing spaces of engagement for teachers, with teacher leadership and some support from university facilitators. Our focus in both phases of the DIPIP project has been on teachers learning to engage with learner errors.

LEARNER ERRORS

In our project we define errors to be systematic, persistent and pervasive mistakes performed by learners across a range of contexts (Nesher, 1987). We, together with others, distinguish errors from slips (Olivier, 1996), which are mistakes that are easily corrected when pointed out. Since errors are systematic and persistent, they are not necessarily responsive to easy correction or re-explanation of concepts, as many teachers know. An important question for our project, drawn from the mathematics education research literature, is how to engage with errors without blaming learners or teachers.

Teachers can evaluate errors in different ways. One way is to avoid errors, which may arise from teacher concerns about judging or shaming learners, or a fear that bringing errors into the public realm will support a "spread" of errors among learners and create more obstacles and stumbling blocks. This approach ignores the need for learners to gain access to appropriate mathematical knowledge. A second way is to correct errors. This makes the appropriate knowledge available to learners but depending on how the errors are corrected, may not illuminate the criteria by which something is judged to be an error, thus still denying access to important mathematical knowledge. A third possibility is to embrace errors (Swan, 2001) as a point of contact with learners' thinking and as points of conversation, which can generate discussions about mathematical ideas. In this way learners' thinking and mathematical knowledge are brought into contact with each other.

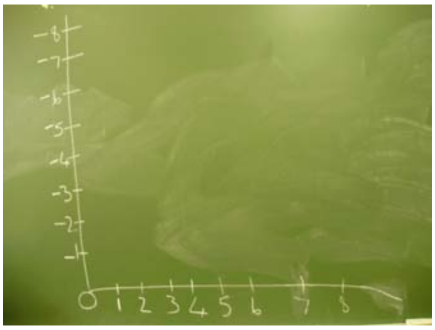
The DIPIP project works with the third option above. A key theoretical consequence of this position is that errors are a normal part of the learning process, for both old-timers and newcomers (Smith, DiSessa, & Roschelle, 1993). Even experienced mathematicians make errors and in so doing create new knowledge in mathematics, thus recreating and shifting the boundaries of the practice (Borasi, 1994). The key point here is that errors are reasonable, make sense to the person who makes the error and are part of gaining access to mathematics and developing it further. So errors make for points of engagement with current knowledge. This notion of errors gives us a way to help teachers to see learners as reasoning and reasonable thinkers and the practice of mathematics as reasoned and reasonable (Ball & Bass, 2003). If teachers search for ways to understand why learners may have made errors, they may come to value learners' thinking and find ways to engage their current knowledge in order to create new knowledge.

LEARNING ABOUT LEARNER ERRORS IN PROFESSIONAL LEARNING COMMUNITIES

In the rest of this paper, I provide two examples of teachers' learning in professional learning communities, one from Phase 2 of the DIPIP project and one from the pilot of Phase 3. In each of these examples I show how the teachers came to a stronger understanding of data about learner errors, informed by the discussion in the community. I also show that certain characteristics of professional learning communities are evident in the data and that these support the teachers' learning.

Example 1: The Numberline and the Cartesian plane

The first example comes from a Grade 9 classroom where the teacher was working with the learners on the Cartesian plane. He called up a learner to label the axes of the first quadrant and the learner drew the following:



The teacher, Tebogo[1], presented this to a group of Grade 7-9 teachers[2]. He acknowledged that he was taken by surprise at this error. The facilitator, Kathy, also acknowledged surprise and said that she had never seen such an error before. At this point, two grade 7 teachers, Nadine and Tarryn made suggestions as to why the learner could have made such an error:

Nadine: I think that the kids are understanding that the numberline has no starting and end point and they know that there are positive and negative numbers, so when it goes from nought to twenty or whatever, then they know that after zero comes negative numbers, minus one, minus two and so on, so they going minus one, minus two, minus three but regardless that its supposed to be a straight line like this (Tarryn nods), its just the lines, these are the positive ones, (indicates horizontally) and these are the negative ones, (indicates vertically), so its just, it's a line

Kathy: Ohhh, so that one becomes flipped over that way

Nadine: Ja (yes)

Tarryn: Ja (yes) …

Tarryn: I'm nodding my head because I had a grade 7 who did that last year and when I asked him about it, he said, mam you told me zero must be in the middle between the positive and the negative, and he had positive numbers (indicates horizontally), a zero, and then he had negative numbers (indicates vertically), so the way that I have explained it about zero being in the middle between the positive and the negative, I obviously didn't make it clear enough that it must be on one straight, horizontal, one line, as far as he's concerned, it is still a line, its just got a bend in it.

A key element of Nadine and Tarryn's explanations are that they present a case for why the learner's error made sense to him. This is an important part of our project and for every error we see, we ask the question, how can this error be sensible and reasonable from the learner's perspective. Nadine explained what the learner did understand about numberlines, that there are no end points, that there are positive and negative numbers and that the negative numbers precede the positive numbers and zero. She also explained what he might not understand, that it had to be a straight line, from his perspective a bend in the line seemed acceptable. Tarryn confirmed that a learner had provided a similar explanation to her. The assumption, stated by Tarryn and shared among the three of us is that such an explanation is sensible, although incorrect, and we could see the sense in the learner's thinking. Tarryn took the discussion a step further, arguing that she had not explained clearly enough that axes need to be straight lines and thus suggesting a role for the teacher in dealing with errors.

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[1] All teachers' and facilitators' names in this paper are pseudonyms.

[2] A design feature of DIPIP Phase 1 and 2 was that teachers worked in both small Grade-level groups and then presented their thinking to larger cross-grade level groups (see Brodie & Shalem, in press for more detail).

A few turns later, Jacob again raised issue of the teachers' role in sustaining errors:

Jacob: Besides that I think again that to the learner, when we introduced the concept of the Cartesian plane, it was not well explained to them about the quadrants, which quadrant, which axis is positive and which axis is in which quadrant, just like that, or may be it was, may be I've forgotten but it can be both the learners' mistakes and it can also be on the side of the educator again

Here Jacob makes a common move that we see once teachers stop blaming learners for errors. If we understand learners' errors to be reasonable and reasoned, then whose "fault" are they; surely they must be the fault of the teacher for not explaining clearly enough. Both Jacob and Tarryn took this position and thought they could have done a better job explaining the concepts. At this point the facilitator took the teachers' contributions and developed a stronger explanation for the learners' error, drawing on her knowledge of the literature (Borasi, 1996; Nesher, 1987; Smith, et al., 1993). First she argued that the learner could be over-generalising his knowledge of numberlines into the new context of Cartesian planes:

Kathy: Okay, we've got a word for that, and its not that you didn't make it clear enough, of course you made it clear enough, he has over-generalised, because he knows about it this way, now it goes that way, well it makes perfect sense, its an over-generalisation, that's how we think as human beings, its not his fault, its not your fault, it's the way we think …

Later she argued that the learner might be struggling to work with two competing explanations:

Kathy: Because in grade 7 the teacher could have explained very nicely about the numberline, in grade 8 or grade 9 the teacher could have explained very nicely about the Cartesian plane, the Cartesian plane with the four quadrants, now the teacher draws only half of each if you want, one quadrant, so the kid's now got two competing explanations, the number line, which flips, or a quarter of the Cartesian plane, which one does he choose, maybe the one that he learned first (people nod and smile) so its no-ones fault, that's the whole thing with these misconceptions, its no-ones fault, it just happens in the process of learning, and this is a fantastic example, I haven't seen it in the literature, you know, it's a new one that maybe we've discovered here.

Kathy makes two key points here: 1. That the learner over-generalises his knowledge of the numberline to the new context and instead of seeing the Cartesian plane as the intersection of two straight numberlines, the learner imagines that one numberline is bent around zero and 2. the learner's error does not necessarily arise from poor explanations by teachers, they are a normal part of the learning process and can come about because a learner is struggling to reconcile two correct understandings. Furthermore, although the error is no-ones "fault", if we understand the source of the error we may be in a better position to correct it.

In the above conversation we see a number of indications that a professional learning community is functioning well. First we see a "challenging focus", trying to explain particular errors that we might not understand from the learner's perspective. The focus comes from real classroom data, so teachers are motivated to understand the error and understanding it is likely to help in similar situations in the future. Second, we see both the teacher and the facilitator willing to acknowledge that they do not understand the error and two primary school teachers able to explain it, given their knowledge of learners. A key element for the success of professional learning communities is to be able to admit to one's own difficulties in understanding learning and to be able to learn from the knowledge of others. All members of the community need to be able to do this, including the facilitator. This does not take away from the facilitator's role in providing expertise. Third, we see that the facilitator is able to link the discussion to the broader research literature thus building rigorous inquiry in the community and creating possibilities for further learning. The understanding that is built of the learner's error and the reasons for it are collective, a number of group members contributed in various ways and this conversation could not have happened without each of the participant's contributions, so we see a collective sense of inquiry into something unknown and a collaborative building of further knowledge, led by the facilitator.

Example 2: Decimal Fractions

This example comes from the test analysis activity. The learners had been asked to express 5/8 as a decimal fraction and had given various answers such as 5,8; 0,85 and 0,58, and when asked to express 5/8 as a percentage, had given answers such as 58%, 85% and 5.8%. The facilitator, Sibanda, had asked about the similarities between these errors and the following conversation took place about the first question:

Nozipho:	Decimals means he knows or she knows that we have a nu…a number before comma and a number after, so what she said or what she writes just add zero and comma because…
Mandla:	No I think…
Nozipho:	…and gets zero comma five eight
Sibanda:	Yes but its now five eight…
Mandla:	*Inaudible*
Nozipho:	…how can I do
Sibanda:	But if you look at the digits that appear in their answers there
Mandla:	They are the same…
Tsepo:	Ja
Mandla:	…they are still there
Sibanda:	Who…what is still there
Mandla:	The very same number, but there are problems … as the that one denominator is numerator
Sibanda:	Where one of them has zero point eight, that's right, zero point five eight, the other has zero point eight five
Mandla:	Aahm
Sibanda:	But the bigger observation as Mandla was saying…
Mandla:	Ehm that one that is numerator in the expression and eight is the denominator

Chorus:	Uuhm
Mandla:	The learner just got rid of this sign, the division sign…
Sibanda:	Uuhm
Mandla:	Then he puts…
Sibanda:	Play around…
Mandla:	Ja
Sibanda:	…play around with five and eight…
Chorus:	Ja

In the above transcript, we see the teachers co-produce an explanation for the learners' errors together with the facilitator. First, Nozipho states what the learners do understand, the form of a decimal number, that it has a comma separating two or more numbers. Sibanda focuses the teachers' attention onto the actual digits that the learners have written, various combinations of 5 and 8. Mandla then builds an explanation that the learners have merely used the numbers in the original fraction, they have gotten rid of the division sign in the fraction and then they "play around" with the numbers and insert the commas where they wish.

After some discussion on the answers for the percentage, the facilitator asks "what is the bigger misunderstanding here" and the following conversation takes place:

Tsepo:	The…I think there is the belief that ah that five over eight has to be represented in its equivalent form because now still we have the ah five comma eight…
Sibanda:	So their belief is that…
Tsepo:	Five over eight has to be equally represented in its equivalent form
Sibanda:	The digits…
Tsepo:	Ja
Sibanda:	…five and eight must not disappear
Tsepo:	Ja
Mandla:	Uuhm (*laughter from all)* all this
Sibanda:	That's the bigger, the big misconception, when you are converting a number whether to a fraction whether to decimal fraction or to percentages, the original digits simply cannot disappear

Here the facilitator re-voices Tsepo's contribution ands shifts it to a claim that the learners believe that the numbers in the original fraction must somehow remain in the decimal. This explanation is powerful in that it explains all of the errors in both parts of the question, in the expressing of the common fraction as both a decimal and a percentage. The facilitator then continued to challenge the teachers as to why learners might make such an error. He suggested to them that it might be because when they first teach conversion from fractions to decimals and percents they use examples with multiples of 10 in the denominator, and in those cases the numerator remains in the decimal. He argued that the use of easier examples creates the possibility of over-generalising and asked the teachers what the key issue of this is for their teaching:

Sibanda:	So as teachers we may go for such easy examples right…but they create a misconception in the learners…
Chorus:	Uuhm
Sibanda:	…right, and the misconception is that the learners begin to think that oh when I am divi…converting to a decimal fraction then the digits…
Tsepo:	Must not disappear
Sibanda:	…they don't disappear…and this is all because of that…
Chorus	Uuhm
Sibanda:	…its because of that…so the big misconception of the digits must not disappear…so the critical issue is how do we…what are…for us as teachers what is the critical issue there
Mapula:	For us teachers…
Sibanda:	Uhm
Tsepo:	Eh I think eh as as we are teachers maybe we should take this example and another example where they are disappearing like that ***(referring to 5/8)*** maybe they will see the difference
Sibanda:	All we are saying is…
Mandla:	According to this one eh one must change examples when teaching

Here we see two teachers, Tsepo and Mandla arguing that when they teach they should use different and more complex examples, rather than the easier ones that they tend to use.

This episode shows a different kind of facilitation in the professional learning community, which also creates a collective engagement and inquiry into learners' errors. Again, we see a teacher explaining what the learners did and then with some prodding by the facilitator, why they made the error and why it might seem reasonable to them. In this case, the reasonableness in the error was seen as more directly linked to the ways in which teachers teach, and so the teachers were supported to think in new ways about their teaching practices and how these might produce errors. Again, we see teachers willing to acknowledge weaknesses in their practices, to refrain from blaming learners for their errors, to critique their own teaching and to find ways to improve their practices. We also see a community where various voices and positions come together to build a stronger, collective understanding of errors in mathematics and the reasons for them.

CONCLUSIONS

One of the key principles of the DIPIP project is that in coming to understand learner needs, teachers can come to understand their own learning needs and how to challenge and improve their practice. The two examples in this paper, and we have many others, show teachers coming to understand learner errors and the reasons for and reasoning behind these. Even though we see slightly different styles of facilitation[3], in both examples we see a common focus on understanding why the learners made the errors, what could they have been thinking, or what were they doing. These are questions that we pose a number of times in our work with the communities and we also aim for teachers to start asking themselves these questions, as they take on the work of leading the community.

The next set of questions that we ask relates to how the community takes these understandings further. The new work includes developing lessons that engage with the identified errors, for example working with more difficult fractions in example 2, thinking about how teaching can produce and reproduce errors, as in example 1, and thinking about how different mathematical explanations come together to produce and reproduce errors, as in both examples. So while we do not argue that teaching "causes" errors in any simple way, we do argue that teachers can work with, engage and transform errors into appropriate mathematical knowledge.

A key element of data-informed professional learning communities that I have discussed through the above examples is the collective nature of the work. We look at data from a particular classroom or across a set of responses to a test and using this data, the community builds explanations and information about the error(s). This information becomes a point of challenge to current understanding and practice and thus a point of departure for rigorous inquiry and possible shifts in knowledge and practice.

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[3] It is not the aim of this paper to discuss these differences more fully but we note that they could be due to the different styles of individual facilitators or to the stage of development of these communities: the first example comes towards the end of the second year of this community working together while the second comes from the first few months.

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THE ROLE OF HISTORICAL DEVELOPMENTS IN CALCULUS

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Calculus is the core of any undergraduate Mathematics curriculum. Not only does the course provide gateway to other higher Mathematics courses, it is also a required course to the study of other physical and engineering sciences. There have been several studies on how the course is taught and learned. In this paper we focus on the efforts that consider historical developments. We believe that the historical development of a Mathematical topic, in particular the difficulties that were encountered and obstacles that were overcome, can tell us something about how an individual might learn or fail to learn that topic. We follow the historical developments of calculus through distinct periods .We will highlight the achievement(s) of each period including the problems that were being solved during that period and what might benefit students.

INTRODUCTION

The importance of calculus in a Mathematics undergraduate curriculum is well documented. Not only is the course the gateway to higher level Mathematics courses, it is also fundamental to the study of other physical sciences and engineering. Kleiner (2001) believes that “the invention of calculus is one of the great intellectual achievements of civilization”. He asserts that the fact that calculus has given precise (mathematical) expression to such fundamental concepts as motion, continuity, variability, and the infinite, Physics and modern technology would be impossible without it; an assertion supported by the fact that most important equations of mechanics and the physical sciences in general are differential and integral equations.

It is precisely due to the importance of calculus that discourses emerged around how the course is taught and how students learn and perform. These include the Calculus Reform initiatives and recommendations that followed. Of interest to the author is the discourse that focuses on the role of historical developments in the learning and teaching of the course. In particular, the concept that the historical development of a Mathematical topic, in particular the difficulties that were encountered and obstacles that were overcome, can tell us something about how an individual might learn or fail to learn that topic. There are a few studies that focus on the interplay between historical developments and the teaching of Mathematics: Kaput (1994), Bressoud (1994), Stahl (1996), Tall (1990), Katz (2000) just to name a few.

In this paper, we will follow the historical development of calculus through distinct periods .We will highlight the achievement(s) of each period including the problems that were being solved during that period. We believe that such a journey provides opportunities for providing students a view of the historical development of a concept; a view that we believe might enhance the understanding of the concept. It should be noted that the treatment of the historical developments is by no means exhaustive.

HISTORICAL DEVELOPMENTS

The historical developments of calculus have been associated with certain distinct periods. Kaput (1994) identifies these periods as the roots of calculus. We will follow Kaput's identification of these periods. The following are the roots as suggested by Kaput:

a. Geometric issues related to computations of areas, volumes.
b. Continuous variation of physical quantities
c. Inherently theoretical concerns with the foundation of Calculus

Geometric issues related to computations of areas, volumes

During this era (15th – early 17th century), "a major (geometric) tool for the investigation of calculus problems was the notion of an indivisible" (Kleiner 2001). It was Cavalieri, as Kleiner (2001) notes, who in his "Geometry of Indivisibles" of 1635 shaped the concept of indivisibles onto a useful tool for the computation of areas and volumes. In this book, the Italian mathematician used what is now known as Cavalieri's Principle: If two solids have equal altitudes, and if sections made by planes parallel to the bases and at equal distances from them are always in a given ratio, then the volumes of the solids are also in this ratio. There is an area version of the principle: Suppose two two-dimensional regions A and B have equal cross sectional lengths for all horizontal lines. Then, A and B have the same area. A more powerful version can be stated as follows: Suppose A and B are two two-dimensional regions and k is a constant such that, for every horizontal line, the length of the cross section of B is k times the length of the cross section of A. Then the area of B is k times the area of A. This can be used to prove that the ellipse defined by $({}^{x}/_{a})^2 + (y/b)^2 = 1$ has area πab.

It should be noted that whereas Cavalieri's method received a lot of criticism for its lack of rigorous proof, real analysis can now be used to prove his method.

Further, noting that we now have the tools of integral calculus to calculate areas and volumes, it is in the author's opinion, instructive for students to note the evolution of the tools. In particular, Cavalieri's method treated a plane region as being made up of infinitely many parallel lines, each considered to be an infinitesimally thin rectangle and an indivisible part of the region (since its width cannot be further subdivided); the area of the region therefore consisted of "all the lines", as Cavalieri asserted. This can be contrasted with the area under the curve in integral calculus.

Continuous variation of physical quantities

Whereas the era discussed above was concerned with investigation of individual objects such as area, volume, and tangents, this era (mid 17th century) is concerned with variation of these objects, or certain qualities of them such as position and velocity. Dubinsky (1994) argues that the main step in passing from the focus on individual objects to the variation of these objects with reference to a particular quality of an object is the encapsulation of that quality. He argues, "Only by thinking of something as an object is it possible to compare it with something else, or itself at a different time" (p 5). The major actors of this era are Newton and Leibniz who are considered as inventors of calculus.

In terms of achievements of this era, the following characterized the achievements of this era: (adapted from Kleiner, 2001 page 142)

a. Invention of the general concepts of derivative and integral: Ad hoc methods that were used to calculate areas and volumes of solids are subsumed under a single concept of the integral.
b. Recognition of differentiation and integration as inverse functions: The development of the Fundamental Theorem of Calculus
c. Development of notations and algorithms
d. Extension of the range of applicability of methods of calculus: Techniques applied to all functions, algebraic and transcendental as opposed to earlier applications that tended to focus mainly on polynomials of low degree

As Kleiner notes (p 144), it is important to note that the calculus of Newton (and of Leibniz) is a calculus of variables; it is not a calculus of functions. In fact, Kleiner emphasizes, the notion of function as an explicit mathematical concept arose only in the early 18th century. Further, it is also important to note what the historian K. Pederson (1980) observed in terms of the achievements of this era:

> An important reason why mathematicians [of the early 17th century] failed to see the general perspectives inherent in their various methods [for solving calculus problems] was probably the fact that to a great extent they expressed themselves in ordinary language without any special notation and so found it difficult to formulate the connections between the problems they dealt with.

In the author's opinion, importance of notation as Pederson notes, should be highlighted to our students to allow for the appreciation of the formalism of the concepts and hence the applications in several problems. As Leibniz notes:

> In signs one observes an advantage in discovery which is greatest when they express the exact nature of a thing briefly, and, as it were picture it; and indeed the labour of thought is wonderfully diminished. (Leibniz quoted in Cajori, 1929, p.184)

Inherently Theoretical concerns with the foundation of Calculus

This era (from mid 18th century) is characterized by a movement to put the principles of calculus on a firm logical foundation propelled by a move towards formalism. We see Euler around the mid- 18th century providing a transition from calculus of equations to calculus of functions. We see the emergence of the notion of a limit as the underlying concept of calculus. We also see the formal definition of continuity emerging. In particular we see the move towards rigorous methods as reflected in the works of Cauchy, Bolzano, and Weirstrass.

The distinguishing foundational features as outlined by Kleiner (2001) are as follows:

a. The emergence of the notion of limit as the underlying concept of calculus;
b. The recognition of the important role played by inequalities and proofs
c. The acknowledgement that the validity of results in calculus must take into account questions of the domain of definition of a function
d. The linking the logical foundation of calculus to the understanding of the nature of the real number system

This is the era which gave us the body of calculus as we know it today and the inherent rigor. This is also an era replete with examples of development of definitions and proofs. In the author's opinion, students must be shown the developmental trajectory of definitions in an attempt to create an understanding of the definitions. As Kleiner notes:

> To begin a calculus course with a definition of a limit may be logically constructive but pedagogically destructive. In general, rigor for rigor's sake will defeat the students. They must be convinced of the usefulness, the importance of having rigorous definitions and proofs..... (Kleiner 2001, p 166)

CONCLUSION

From the discussions above, we note that each era provides unique developments whose discussions with students can provide an evolutionary explanation of today's methods in calculus. Whilst we do this, however, it becomes important to acknowledge that while each new development appears progressive in relation to its predecessor, "this perception of progress is a local and not necessarily a global characterization" (Confrey and Smith 1994). To generate an appreciation of an evolution of a concept over a historical period, it is important to treat contributions of each period as equally valid and to understand the viability of the period. As Confrey and Smith (1994) assert, "as we look backwards, it is often easy to see what we now understand but which those before us seemingly did not." They go further to assert that "if we do not seriously attempt to understand the viability of the worlds of those who came before us and to seek out their alternative perspectives, we are too easily led to a self-reinforcing progressive version of history" (p178).

The history of calculus is the history of attempts spanning over 2000 years focussing on variability and continuity. It is a history capturing developments of rigor and formalism. It is a history of progressive abstraction. A history defined as “intellectual accomplishment of the first rank” by others. As educators it is a history whose developments must be shared in our classrooms.

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PLENARY PANEL PAPERS

AMESA'S ROLE IN RELATION TO RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION: CONVERSATIONS WITH PAST PRESIDENTS AMESA 'HOT TOPICS' PANEL DISCUSSION – 2011

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This panel was set up as a forum for taking stock of AMESA's achievements in the post-apartheid era, and for raising questions about AMESA's future directions in relation to research and development in the field of mathematics education. The focus is set in a context of multiple, ongoing challenges – poor performance in mathematics across all phases, evidence of gaps in teachers' content knowledge implicated within this poor performance, and the ongoing presence of inequity of provision. All post 1994 past-presidents of AMESA were invited to share their views on past achievements and current challenges. Four of the five past-presidents were able to be present. In this introduction to their contributions, I outline the questions that were set to frame their inputs, and overview some of their key suggestions.

INTRODUCTION

This panel was set up as a forum in which the AMESA community could engage with the views of past presidents on the role of the organization in relation to supporting research and development in the field of mathematics education. Supporting research and development in the field are central facets of AMESA's constitution. In a context of ongoing reports of a 'crisis' (Fleisch, 2008; Taylor, 2006) in mathematics education, the intention was to stimulate debate on past achievements, current challenges, and future priorities and directions for AMESA.

GUIDING QUESTIONS

The following 'guide' questions were set to stimulate thinking in relation to these issues:

- What do you consider to be AMESA's key achievements during the post-apartheid years?
- What do you see as the key priorities for the next 5 – 10 years?
- What do you see as the opportunities and the constraints in AMESA's structures and practices in terms of contributing to the development of the maths education field?

- How do you view AMESA's role in relation to supporting a profession that works in research/evidence-informed ways?

These questions were selected in order to allow a location of current challenges within a historical trajectory – a trajectory within which all of the panelists have been key players within their roles as past-presidents. Asking the questions from an AMESA perspective, rather than more broadly within mathematics education, was also intended to open up spaces for the audience to hear about some of the inner workings of AMESA, and the possibilities and constraints to impacting on the field within its structures and practices.

The past-presidents contributing to this 'Hot Topics' discussion bring varied experience from the mathematics education field – as mathematics teachers, teacher development leaders, curriculum and policy contributors, and academics. They represent in many ways, the broad church that makes up the AMESA community, and one of the relatively small number of organizations, nationally and internationally, that explicitly seek to bring these sometimes disparate communities together. This broad church provides an important forum for sharing information, rationales and debate, and yet, there is a sense that the potential offered by this space has not been optimally drawn upon to contribute at national, systemic levels to improving mathematics teaching and learning in our public school system. This in spite of the fact, noted in several of the past presidents' papers, that AMESA has been highly successful in growing its membership and conference attendance base over the last decade or so.

All four panelists' papers offer suggestions in relation to potential foci for AMESA in terms of contributing to this space: Duba and Mogamberry (both papers in this issue) focus on the urgent need for teacher education; Brombacher and Setati (both papers in this issue) comment on the need for AMESA to offer a profession-orientated, research informed lead on key issues (Brombacher mentions calculator use/ curriculum and assessment changes as examples) through the production of position statements and commentaries. Setati (ibid) also raises the need for AMESA to play an instrumental role in identifying 'gaps' in research findings – using its grounded strength across multiple communities with interests in mathematics education to set the research agenda, an agenda that university departments could use in thinking about postgraduate research opportunities. Interesting overlaps and contrasts are evident across all the papers in terms of thinking about AMESA's interactions with government – should AMESA be following government priorities and directions for mathematics education, or should it take a more critical position in terms of contributing to these priorities and directions, including adopting a position of critique on some issues (whilst acknowledging the shared interest in improving the quality of mathematics teaching and learning) – a 'critical friend' position (Costa & Kallick, 1993) rather than a 'faithful implementer' position? Costa & Kallick (p49) describe the 'critical friend' role in the following terms:

> a trusted person who asks provocative questions, provides data to be examined

> through another lens, and offers critique of a person's work as a friend. A critical friend takes the time to fully understand the context of the work presented and the outcomes that the person or group is working toward. The friend is an advocate for the success of the work.

I take this opportunity to thank all four past-presidents for their willingness to contribute to this 'Hot Topics' panel discussion, through sharing their insights and opening up their views to robust debate.

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TOWARDS A RESEARCH AND DEVELOPMENT AGENDA FOR MATHEMATICS EDUCATION IN SOUTH AFRICA

Mamokgethi Setati

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AMESA Past President (2002 – 2006)

During the AMESA National Congress in 2003 when we celebrated our ten-year anniversary I gave a talk as the then National President of AMESA, in which I reflected on AMESA's past, present and future (see Setati, 2003). On that occasion I focused on AMESA's challenges and achievements in the first ten years of its existence by reflecting on who we are, who we have been, who we want to be and who we should be as a mathematics education professional association in South Africa. It is now eight years since that reflection and 18 years of AMESA's existence and my view is that while we may be clear about who we are and who we have been, the questions about who we want to be and who we should be remain important. In this paper I present an argument for AMESA to have a research and development agenda for mathematics education in South Africa.

PANEL ADDRESS

The AMESA constitution states, "The aims of the Association shall be, in general, to promote Mathematics education and, in particular, to enhance the quality of the teaching and learning of Mathematics in South Africa". The Constitution further states that the Association envisages attaining the main aims by encouraging research; formulating policy statements on matters regarding mathematics education, encouraging excellence in mathematics education and actively engaging in projects that will result in the social, economic, political and cultural development of society. The question is whether we are delivering on these aims and strategies listed in our constitution. For example, when last did AMESA formulate a policy statement on any matter regarding mathematics education? How is AMESA attending to redress in our society? More importantly, the question to ask is how can we know whether we are delivering on the aims and strategies listed in our constitution?

There is no doubt that the number of educators attending the annual AMESA Congresses has increased to more than a thousand in the past few years and so has the membership. But does this mean we are doing a good job? In my view this rise in numbers does not tell us much about the quality of our work as an Association. It also does not tell us whether we are taking leadership in South African mathematics education as is required of an association such as AMESA. We need to have new or different measures for our success. We need to start asking questions about the quality of our congresses (both in content and form); the quality of our research, development and advocacy work and the quality of leadership we offer the South

African mathematics education community.

Despite the very welcome growth in the number of delegates at AMESA National Congresses, I remain concerned about the fact that the program is not necessarily growing both in quantity and quality. What worries me about AMESA Congresses is that a large number of people attending the congresses do not present papers. There is a worrying tendency to come to the Congress to receive rather than to give. I know that some people think others at the conference know more than them and so they just come to listen and not to contribute. As an association we should be concerned that even some who are in the leadership of AMESA do not present papers at Congresses. In my view this is the reason why nobody is asking critical questions about why an increase in number of people attending the congress is not translating into an increase in the number and/or quality of papers presented. Nobody is asking the crucial questions about who presents papers at congresses and who is not presenting and why? During the period 2000 - 2006, we used to hold first time presenters workshops in Gauteng to assist people who have never presented papers at the AMESA congress. These people were allocated mentors who assisted them with the preparation of their papers. We even raised funds to ensure that the people who attend the first presenters' workshop and end up having a paper accepted attend the congress.

I know that some of the questions I am asking above are somewhat provocative, but the truth is that if we are going to deliver on the aims and strategies listed in our constitution then we have to ask the questions that I am raising above and then sincerely work together to find solutions. No doubt there are several in-house issues that must be dealt with in AMESA and more of those can only be dealt with if the leadership is open to ideas from outside itself. My hope is that an opportunity will be created for a productive engagement with former presidents on the future of AMESA. At this point I want to focus on the role of AMESA in providing leadership for mathematics education in South Africa.

Besides its weaknesses, AMESA remains a leading professional association for mathematics teachers, teacher educators and researchers in South Africa. AMESA members are often called upon (oftentimes as individuals, which is unfortunate) to identify research and development priorities for South African mathematics education. Given its positioning and role, it would be helpful for AMESA to develop an accessible and flexible research and development agenda that lists high-priority research questions and development issues that the field of Mathematics education in South Africa would like to see answered. Such an agenda would be revised regularly and made available to multiple audiences. Graduate students and research supervisors could consult the document to identify possible research topics. When judging proposals for requests for funding, decision makers such as funding agencies could consider the document to help them select projects and therefore make effective use of limited resources. When considering major policy decisions government departments could consult the agenda to see what the key priority issues are in South

African mathematics education. Policy makers need timely and accurate information about mathematics learning and teaching to inform their decisions. The research and development agenda can serve as an important tool to familiarize them with questions still needing to be answered and thus help prioritize future projects and policy initiatives. The research agenda would thus help organize and coordinate research and development in mathematics education in South Africa.

My proposal therefore is for the AMESA National Council to form a Research and Development Agenda Task Team that can produce a research and development agenda for Mathematics Education in South Africa. Such an agenda would highlight the most pressing questions and issues for mathematics education in South Africa. It is of course obvious that research and development areas of importance today will not necessarily be of importance years from now, and the agenda would need to take this reality into consideration by indicating a mechanism for regularly reviewing and modifying the agenda.

Colleagues, South African Mathematics education remains in crisis and we as members of the profession must be concerned. You all know the problems that we face in mathematics education in South Africa. They include poor matric results nationally in Mathematics as well as very small numbers of African candidates obtaining university exemption in mathematics. Even when we have an increase in the number of students passing mathematics at matric level like we did in 2008, we remain concerned about the quality of the mathematics knowledge that our students come out of high school with. We are concerned about this situation because it has dire consequences for our company in particular but also for the development of our country in general. As a leading mathematics education association in the country we need to realise that unless we increase the quality and quantity of students who can become the next generation of scientists, engineers and technical specialists, South Africa's vision for a sustainable democracy will not come to fruition. So I think it is time for us as an association to re-think our strategy, by thinking critically about what we do and how we do it. What does it benefit us to come to the Congress every year and then go back to our schools or institutions and face the very same problems without any idea of what the possible solutions could be? What is the purpose of AMESA and why have a Congress?

As an association we need to realize that our challenge is not just to make sure that more teachers attend the AMESA Congress or take more teachers through mathematics development programs or just to get more learners to pass matric mathematics. Our challenge as a leading mathematics education association in the country is to solve the problem that our country is currently facing in mathematics education and my view is that a research and development agenda can serve as a roadmap towards solutions. It is our responsibility to provide leadership to the South African mathematics education community on what needs to be done to solve the problems that we are facing.

In talking about this enormous problem that we have in front of us and the need for us as an association to solve it, I want to recall an event that happened 42 years ago in 1969 when human beings set forth, sitting atop the Saturn V rocket, for the first time in an attempt to set foot on another piece of rock other than our earth. It was not an easy task and I am sure there are many at that time who thought, “why do the impossible!?” I am reminded of this event because the responsibility that we have as an association is not an easy one and given the concern I raised earlier in this paper, I am sure there are many AMESA members who think finding a sustainable solution to the crisis in mathematics education is not something they can do or want to do. In his address delivered on 12 September 1962, during the launch of the Nation's Space Program at Rice University, John F. Kennedy said,

> We choose to go to the moon in this decade and do the other things, not because they are easy, but because they are hard, because that goal will serve to organize and measure the best of our energies and skills, because that challenge is one that we are willing to accept, one we are unwilling to postpone, and one which we intend to win, and the others, too.

And so I want use the same words to challenge us as an association as we approach AMESA’s 20th anniversary in 2013: We, as AMESA, must choose to attend to the crisis in mathematics education in South Africa not because it is an easy problem to solve, but because it is hard, because the goal will serve to organise and measure the best of our energies and skills in mathematics and mathematics education, because this is the challenge we are willing to accept, one we are unwilling to postpone, and one which we intend to win.

Colleagues, what we are faced with is a crisis and I know that many of us South Africans do not like using the word ‘crisis’. The Chinese use two brush strokes to write the word "crisis". One brush stroke stands for danger; the other for opportunity. In a crisis, we have to be aware of the danger — but we also have to recognize the opportunity. As AMESA we have to be attracted to the idea of solving the problem because of the opportunity it presents us.

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AMESA'S ROLE IN RELATION TO RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION: CONVERSATIONS WITH PAST PRESIDENTS

Ray Duba

Teacher INSET – Palabora Foundation

AMESA Past President: 2006 – 2008

I like the punch line of the SABC TV Weather presentation. It reads "You get to understand the climate through the weather" I am quite sure that Hamsa Venkat, the chair of this panel discussion, noted the AMESA "climate" that left questions in her mind; hence this topic. What is the weather in our Association like that determined the current climate? I would like to paint the current climate as I see it in this paper for the live panel discussion.

GUIDING QUESTIONS FOR THIS PANEL "HOT TOPIC" DISCUSSION

1 What do you consider to be AMESA's key achievements during the post-apartheid years?
2 What do you see as the key priorities for the next 5 – 10 years?
3 What do you see as the opportunities and the constraints in AMESA's structures and practices in terms of contributing to the development of the maths education field?
4 How do you view AMESA's role in relation to supporting a profession that works in research/evidence-informed ways?

The objective of this paper is not to present a definitive answer to these questions, but to start or continue a debate on what role AMESA (or any other association with similar objectives as that of AMESA) should be playing to improve the situation in relation to research and development in the field of mathematics education?

It is important to evaluate issues that I raised in this paper against my background. I am past president of AMESA. I am not a formal researcher and definitely not claiming to be an expert in Mathematics Education. I am heading an INSET programme of a community project, forming part of the corporate social responsibility efforts of a local mine. I am simply raising my opinion to contribute to the debate.

The strength of an association

I have noted with excitement that attendance at conferences has increased from just over four hundred fifteen years ago to over a thousand in 2010. What does this increase tell us? Are teachers hungrier for knowledge and feel that they can get it from AMESA conferences? Or is it because there is more money made available by the government for them to attend? As AMESA do we know what specifically has led to the rise in attendance or are we just pleased to see a rise in attendance? Did we ask this question? Do we know as AMESA who is performing and who is not? Who needs development? Who doesn't? When an international organization called and asked me to forward names of teachers who are AMESA members for overseas training, we did not have this information at hand to send. When the Provincial Department of Education asked for names of teachers to sponsor for the conference, we did not have criteria; we did not readily have names; we did not have a pool of teachers to select from. We kept no database, let alone an up to date one.

AMESA conferences have only benefited researchers from universities by assisting them to comply with the "publish or perish policy" as it is commonly known, rather than it benefiting school teachers. I gathered from some teachers that researchers learn from teachers and write their papers while teachers learn very little or nothing. I also noted the little time teachers spent with researchers in certain projects. Why are most teachers not researching to emulate the researchers? Do researchers go back to the teachers and teach them how to teach in accordance with their findings? I would like to see more transfer of research skills to more teachers.

Going forward, AMESA must conduct a small research project and entitle it "Where are the teachers?" Where are the teachers who attended AMESA organized workshops, such as the DST Educator Support Programme (ESP) funded by DST in 2006? Who is presenting at AMESA conferences? Where are teacher presenters and how are they doing? They should initiate an article to profile teachers in AMESA news. Follow up for them to present again, and provide some kind of a report back on how it was improved over 12 months.

Mandate from the department of education

AMESA is a national asset by virtue of its contribution to Mathematics education research already conducted by our members, but this national asset is underutilized. We are invited to participate in ministerial committees but only a small unrepresentative team from AMESA takes part. To be a real voice, AMESA must strive to get its mandate from the government on educational issues (identified needs) to be researched. In 2006 AMESA established a partnership with the Department of Science and Technology and started the Educator Support Programme. We must identify projects that aim to support the department's goals. The media will pay attention every time there is an AMESA conference such as this, expecting to hear what AMESA pronouncements will be made. The allocation of close to R2 billion for teacher training shows an acknowledgement of the pressing need for prospective teachers in mathematics, science and the foundation phase. The system (apartheid and current) allowed people who are not qualified to teach mathematics. The system hadn't invested in our teachers at all, until about three years ago when bursaries for four year degrees and ACE diplomas were awarded to teachers. The department gives teachers guidelines through the various policy documents; however the guidelines go to the extreme to such an extent that the practice is undermining creativity and innovation. Teachers must start demanding and toyi-toyi for proper Continuing

Professional Teacher Development (CPTD) from their HODs, principals and the department of education. International evidence shows that CPTD succeeds best when teachers themselves are integrally involved, reflecting on their own practice; when there is a strong school-based component; when activities are well co-ordinated; and when employers (and I would say this is employer commitment) provide sustained leadership and support (National Policy Framework for Teacher Education and Development in South Africa, 2007). The framework goes on to say:

> However, it is the responsibility of teachers themselves, guided by their own professional body, SACE**,** (and I would include AMESA) to take charge of their self-development by identifying the areas in which they need to grow professionally, and to use all opportunities made available to them for this purpose, including those provided for in the Integrated Quality Management System (IQMS).

For some reason AMESA is not taking advantage of this opportunity. What is AMESA's research and development budget? In a globalised world, as our conference theme alluded to, only institutions that address societal concerns of both teachers and learners will have a competitive edge. Are we talking to the right directorate of the Department of Education?

Additional outside or non - mathematics expertise

We do not have all the smart people at our disposal to solve our problem. If there is any change in the "crisis", it is too slow. We failed as a community of mathematicians. Let us invite outside people – non mathematics people. Invite a lawyer to our conference alongside a SAMS academic. People across all professional fields have got into their professions through being taught by teachers. In 2004 when I was still chairperson of AMESA Limpopo region, we invited a district senior manager, Mr. WWX Nkuna, of the Mopani District of the department of Education to our regional conference. I would like to share with you what he said on the day:

> My interaction with mathematics started in 1965 and it ended in 1969. Those were my high schools years. I used to enjoy arithmetic but my performance was luke-warm. My view was that mathematics was too demanding. As I ascended high school grades, I found maths more and more difficult. My teachers did little to motivate me. I only continued with mathematics because I had earmarked medicine as a career. All that dissipated the moment the final results were released. You see, I obtained an F symbol in mathematics and maths was a precondition for admission to medicine. I was bound to choose another direction. Indeed I chose another direction and closed the chapter on mathematics in my life. Ever since, maths existed away from my focus.

MATHEMATICS EDUCATORS WERE RIGID IN THEIR APPROACH

> For all this, don't blame me; blame my teachers who failed to make mathematics interesting for me.
>
> - They were extremely rigid.
> - They allowed no deviation except what was in the memo.
> - They mystified mathematics and made it difficult

- They preached mathematics predestination
- In a changing world mathematics was made to look rigid:
  - ✓ That things change is indisputable
  - ✓ An aerorplane on the ground and the same aeroplane in space will never be the same
  - ✓ One kilometre with a lover will never be the same if you are carrying 80kg bag of mealiemeal.
  - ✓ Thirty minutes of entertainment will never be the same with the same amount of time with your finger on fire.

How true is that?

Maths results are never made in FET

Our schools must also plan and allocate work wisely. It is imprudent to assign all the good teachers at exit points and to assign inexperienced and inefficient teachers at entry levels. Results are not made at matric level, but at lower levels. Once a learner can do well in lower grades, then that learner is unstoppable. Every school must work according to a plan, if it will make a mark. Teaching without a plan is effectively cheating. Research and development must begin to focus on the Foundation Phase.

AMESA must begin to be the real voice of mathematics education where it matters most - in the schools. AMESA members must influence, be a watch dog. But currently, we do not have muscle for this.

IN CONCLUSION

The challenges in education and in mathematics education in particular are so overwhelming that I found it difficult to conclude this article. I am glad that this article is only meant to give readers a taste of what one past president's contribution to the hot topic is. More will unfold in a live debate at the planned panel discussion at this conference. We will give you the weather so that you can prepare for the climate ahead.

AMESA'S ROLE IN RELATION TO RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION

Aarnout Brombacher

Brombacher and Associates

AMESA Past President: 1998 - 2002

AMESA is, in pursuing the Aims of the Association, charged by its constitution to encourage "research related to Mathematics education" and to "bringing the results of such research to the attention of its members". The Association is also charged with"formulating policy statements on matters regarding Mathematics education and promoting such perspectives among its members, policy-making bodies and organs in civil society involved in education." In this paper I make the point that the latter of these strategies identified for achieving the Aims of the Association is in need of attention. By looking at some South African mathematics education realities I attempt to show how position statements based on research would enable the Association to make a more authoritative impact.

INTRODUCTION

That AMESA has a role in relation to research and development in mathematics education is undoubted. AMESA not only has a role, but as an organisation it is mandated through its constitution to play such a role. The real question is: How does AMESA meet its obligation in this regard? As a past president of the Association I am very mindful of realities that make meeting this obligation a challenge. The Association is no more than a collection of professionals all with their own jobs and obligations outside of the Association who, with the best wills in the world, contribute to and participate in the activities of the Association for a range of different and often personal reasons. For none of these individuals is the Association their first obligation. The simple reality is/was that, at least during my terms as president, we found it hard to muster the resources to meet the Association's obligation in this regard. This possibly suggests a first research task.

If, as the description of this panel suggests, there is indeed a 'crisis' in performance in Mathematics in South Africa, then I would suggest that we, as and Association, have to examine the extent of our culpability in the crisis. The question I am posing is, whether by struggling to meet our obligation, we are allowing the 'crisis' to persist and/or even contributing to it.

In this short paper I would like to suggest a number of avenues of research that the Association should, in my mind, be pursuing and in closing I will make suggestions on how I think that can be achieved. My remarks should be regarded as personal and reflective in nature.

THE CRISIS AND A PARTICULAR RESPONSE

I will make no attempt, in this paper, to describe “the crisis”. What I will do is to describe a particular response to the crisis and then to ask what AMESA’s role has been and/or could/should have been.

At national (government) level, the crisis in mathematics education is seen as a part of a broader more general ‘crisis’ of underperformance of the education system in general. The recent (2009) elections saw the inevitable changing of the guard in the office of the Minister of Education and together with that, the associated promises of improved performance and delivery. Not long into her term the current minister (as have several before) announced that after a period of listening to, among others, the teachers of South Africa, the country would be embarking on a process of “repackaging” the curriculum. She also announced, in support of the Presidents pledge in his State of the Nation Address of 2010, that as from January 2011 all children in Grades 1 to 6 would receive workbooks for Language (Literacy) and Mathematics (Numeracy).

The “repackaging” of the curriculum is manifesting itself in the so-called CAPS documents which are to be phased into schools as from January 2012 (Grades R to 3 and 10). The first half-year or each workbook arrived in schools in January 2011 – a project with an alleged budget of R2,6 billion. These workbooks were developed in a period of only three months and were never field tested.

The introduction of the CAPS documents brings with it a number of so-called changes (which I will not discuss here) and these changes have necessitated the redesign/redevelopment of textbooks. The deadline for submission of textbooks for screening by the DoBE was 16 June and at the time of writing this paper (April 2011) the CAPS documents were not yet in their final state and publishers and authors were scurrying about trying to write textbooks against incomplete documents.

All of this raises a number of questions: questions that may or may not have answers in research of one form or another.

The first question is whether or not the repackaging of the curriculum without its OBE nomenclature and including a sequencing of topics with recommended times allocated to each topic will make the slightest difference to the “crisis” in education in general and mathematics in particular. This can be researched. That is, a well chosen sample of schools across the provinces can be nominated to work with the first draft of the documents for a few years and progress against key indicators monitored to see if the anticipated improvements are manifesting. Of course associated research would be concerned with identifying the critical factors that contributed to greater success in some schools than in others so that the mass implementation would benefit from this. After the research period, modifications would be made to the hypothesis that motivated the development of the CAPS documents, the documents themselves could be improved and, after a possible second phase of trialing and further improvement, the documents would be introduced to all schools.

That, by the way, is, in broad brush strokes, how medicines move from the laboratory to the shelf over a period of many years. Medicines are subjected to such rigorous research processes because the health of individuals is at stake. We don't perform mass experimentation with cures for TB, HIV/AIDS and cancer, and yet we conduct one mass experiment with the education of our country's youth time after time, minister after minister.

Where is AMESA in this crisis within a crisis? What is or should be the role of AMESA? I am not suggesting that AMESA should be conducting the mass scale research described so far. But I am suggesting that AMESA could be lobbying for such a process. I do not think that AMESA alone will change the course of events but we could be making our voice heard more clearly and loudly and that voice needs to be informed by research – it needs to be informed by evidence.

Over the years, AMESA has a proud record of being invited by the DoBE to play a leadership role in the development of the Mathematics curricula and related documents as it was again in the CAPS process. I have myself played the role of AMESA representative in these processes on more than one occasion over the years. As an Association we have been pleased to be asked to play these roles. On the one hand we have regarded it as recognition of our status as *the* professional subject association. On the other hand we have believed that it is better to be in the process than not. I am starting to question these assumptions.

On the one hand I am worried that by involving the Association, the DoBE buys our tacit support for the process. The Department is left with the luxury of being able to say that they involved the professional association and therefore the product has certain authenticity and implicit buy-in by the broader mathematics community.

On the other hand I am worried that we, as an Association, provide little guidance to and support for our representatives in these roles. I am not asking that representatives be hamstrung in their participation by being forced to seek mandates from the Association on all decisions. What I am asking is that AMESA should develop positions – research based positions – with regard to many of the issues that will face the panels on which the representative(s) serve. In this way the Association would empower the representative(s) to contribute with the research based authority of the Association to discussions on issues such as: the role of the calculator, the importance or not of Euclidean geometry, and what constitutes "the basics" in early numeracy development to name but a few.

Instead when I observe the recent CAPS process and in particular the Mathematics process then I am left with a number of concerns. There was a changing of the guard more than once on more than one of the documents. Some of the documents circulated for public comment were so thoroughly critiqued that the current versions hold little semblance to the documents that were subjected to public comment and yet the revisions are not being subjected to the same comment again and so the story continues.

Let me be perfectly clear about one thing. I am in no way casting aspersions on the *bona fides* of any single person who worked on the teams, AMESA member or not. What I am simply trying to show is that the process was far from ideal and yet the product has the *de facto* stamp of approval of the Association on it. I have a sense that we, as and Association, are overwhelmed by the opportunity of "being involved" and insufficiently critical of the process.

What is the role of research? In mathematics classes we develop among other things problem solving heuristics that broadly involve: understanding the problem; making a plan; carrying out the plan; reflecting on the outcome; modifying the plan in light of the evidence and repeating the cycle until we reach a point where we believe we have a solution to the problem. In our mathematics classes we reject guesses and estimates except, of course, as initial attempts to understand the problem. We reject hypotheses as solutions and demand evidence or proof.

Why are we, as a mathematics education community, as the professional association, not more vigorous in our demand of the same when it comes to mathematics education (and curriculum) in broader terms?

Paper based research of curriculum development processes across the world and in particular in countries that perform well on one of the international scales – the TIMSS study – will show that these countries develop their curriculum in more research based ways; through more iterative processes; over longer periods of time. Would it be an idea for AMESA to research these processes and to publish a position statement on the curriculum development cycle in different countries and the successes and failures of these? In this way we would be able to make research/evidence based comments on the processes taking place in our country and with our children.

THE ROLE OF OPINION VS RESEARCH

Some would argue that the "crisis" in mathematics is easily solved. Having done some mathematics at school makes almost all lay persons believe that they are experts on the teaching of the subject. The problem, I am repeatedly told, is that children no longer have to memorise their multiplication tables, children are allowed to use calculators and we don't teach the basic, whatever those are, properly. The interesting thing is that these opinions are not limited to lay people. AMESA representatives on national examination bodies and AMESA's responses to the National Senior Certificate make similar assertions about the role of the calculator.

How can it be that more than twenty years after the hand held calculator became universally available, that AMESA does not have a more informed position on the use of the calculator as a tool of learning? Why do we allow our responses and our representatives to express opinions as fact on this matter?

Paper based research could involve collating the latest research findings from studies across the world and make these available in a position statement. The research findings in the position statement could in turn underpin AMESA's position on the role of the calculator in particular and on other issues in the teaching and learning of mathematics.

The same applies to the role of multiplication tables – or more broadly the memorisation of number facts, the role of the so called basics and so on. I am not suggesting that the research will produce uniform or definitive answers, but it will take us from opinion to informed discussion.

Now of course this work has already been done and it already exists. The two Handbooks of Mathematics Education (Bishop *et al* 2003, 1996), the various NCTM yearbooks, the research journals of mathematics associations across the world, including our own Pythagoras, and the conference proceedings of innumerable research conferences.

The challenge that I am posing is for AMESA to draw on all of this research, possibly supplemented with commissioned studies in the South African context in cases where local research is limited, and use it to develop Association position statements; position statements that are presented in ways that are easily accessible (readable) to classroom teachers as well as to lay people concerned about and interested in the field.

LARGE SCALE TESTING AND AMESA'S RESEARCH ROLE

There is an emerging trend involving large scale testing of children in South Africa. The DoBE has introduced Annual National Assessments (ANA), the Western Cape (and possibly other provinces) has introduced regular systemic assessments and all this in the interest of monitoring progress and hence improving performance. The powerful backwash effect of testing on classroom teaching and learning is well documented and described (NECC, 1992). Classroom practice is shaped in part by the assessment criteria reflected in the assessments and teachers' understanding of what it means to know and teach mathematics is in turn also impacted on by these assessments. Of course tests in themselves will never improve student performance but if we know that national and provincial testing impact on classroom practice then we would do well to ensure that the tests convey an image of Mathematics that is consistent with an image that the Association holds dear.

Over the years provincial interest groups of the Association have commented on the National Senior Certificate and these comments have in turn had an impact on the nature and form of the examinations. Let us learn from this experience.

The Association would do well to also concern itself with the nature and form of these large scale assessments that are emerging onto the landscape. To do so requires two actions. First and foremost the Association needs to develop a framework for assessing these assessments. What criteria will we use? Will be guided by personal interests and images of what it means to do mathematics or will we draw on research that has determined the knowledge and skills at different age groups which best correlate with future success in mathematics? Does such research exist? And what does success in mathematics mean? What is AMESA's position?

Having established the assessment framework the Association can then apply its energies to evaluating these assessments and providing critique. That said, I also think that an assessment framework for critiquing the National Senior Certificate Examination is long overdue.

IT WORKS FOR ME

I cannot tell you how often I am approached by teachers and others who have the panacea for all our mathematical ills. If only we could get all children to do this or all parents buy that then everybody would be successful in mathematics. In an effort to convince me they will then proceed to tell how their nephew, who used this method, is now an engineer or doctor or, better still, a millionaire.

I am not suggesting that the panacea does not exist – I am working on one myself. The truth, however, is that even if the method or product made a difference in somebody's life, which, by the way, is very hard to prove, then there is no assurance that it will make the same difference in other people's lives. This is what makes our work here so difficult.

It is for this reason that I did not earlier list the AMESA publication *Learning and Teaching Mathematics* or the *How I Teach it Sessions* at a conference like this as sources of evidence in the research gathering that needs to be done. *Learning and Teaching Mathematics*, and *How I Teach it Sessions* provide opportunities for people to share experiences. They provide opportunities for people to say: "this works for me" and then to hear others provide feedback and critique. They start a conversation. This is, however, not research – or at least not research of a sufficiently generalisable form to be helpful in shaping the greater landscape. A lot more work is needed before these ideas can have that impact.

A possible role for AMESA to play in this regard is to help members and others to recognise the difference between research and hearsay. There may also be a role in helping those who are interested in doing more formal research on their product or procedure to find people who can support them through the process in universities and other research institutions in the country.

MAKING IT HAPPEN

In my introduction I hinted that making this all happen is not easy given the nature of an organisation such as AMESA and the profile of its members. If it was easy it would have been done long ago. It is not as if we did not have these discussions in Council when I was president and I am sure it has been on the minds of other Councils and presidents as well.

It is expressly because it is difficult that I have not, at this stage suggested, that we should, as an association, engage in impact research studies at a small or large scale. In fairness, I have to question whether that would ever be the role of AMESA.

What I would, however, propose is that the Association identifies the issues in Mathematics Education in South Africa on which it believes its opinion could have the greatest impact and then to fund (either directly or through donors) the development of research based position statements of the kind that I have already described. I would further propose that the NCTM (and I am sure there are other organisations) has a fairly sophisticated process used in the development of such documents and that it may be wise to draw on their experience and expertise in this regard.

Let me not be naïve and at same time run the risk of being controversial. Funding this work means paying people to do the job. For long, we have as an Association held onto the romantic notion that people work for the Association out of philanthropic zeal. Certainly there is scope for philanthropic work and the continued existence of the Association and its work is evidence that such energy abounds. Developing a significant position statement takes time and expertise – time and expertise that have to be bought.

CONCLUSION

The AMESA Constitution charges us to be concerned with research through both the Aims of the Association and the strategies envisaged for achieving these:

> "The aims of the Association shall be, in general, to promote Mathematics education and, in particular, to enhance the quality of the teaching and learning of Mathematics in South Africa by:
>
> 2.1 Providing a forum for all concerned with the teaching of Mathematics at all levels of education.
>
> …
>
> Strategies
>
> The Association envisages attaining the main aims by the following strategies:
>
> 2.2.1 encouraging research related to Mathematics education and bringing the results of such research to the attention of its members;
>
> 2.2.2 formulating policy statements on matters regarding Mathematics education and promoting such perspectives among its members, policy-making bodies and organs in civil society involved in education;
>
> 2.2.3 actively engaging in mathematics education projects which will result in the social, economic, political and cultural development of society."
>
> (AMESA, last updated 2007)

In terms of the strategies, and research in particular, I think that the Association does better in terms of paragraph 2.2.1 than it does in terms of paragraph 2.2.2. The 2002/2004 National Council made certain that Pythagoras became an accredited journal and in so doing made it more attractive for researchers in the field to publish their research in the journal.

My suggestion is that it is now time to take greater cognizance of the charge in paragraph 2.2.2 and to develop research based policy statements.

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AMESA'S ROLE IN RELATION TO RESEARCH AND DEVELOPMENT IN MATHEMATICS EDUCATION – CONVERSATIONS WITH PAST PRESIDENTS

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AMESA Past President: 1996 - 1998

"Mathematics performance in South Africa continues to be described as being in a state of 'crisis'. What role should AMESA be playing to improve this situation in relation to research and development in the field?"

AMESA'S KEY ACHIEVEMENTS DURING THE POST-APARTHEID YEARS

In July 1993, the launch of AMESA as the only nonracial professional association for mathematics teachers was a major breakthrough for the future of mathematics teaching and learning in post apartheid South Africa. It seems a long while now since the pain and frustration of getting several racially divided associations to shed their racial exclusivity and come together for the formation of a brand new association. I had the privilege of being elected the second national president of AMESA from 1996 to 1998, serving after the first national leader, the illustrious, Dr Mathume Bopape.

The early years were occupied with the difficult task of unifying a deeply divided mathematical community of separate groups each, with different levels of apprehension coming together with different amounts of resources, expertise and expectations. Despite the many tensions members were determined to change the state of mathematics education in South Africa. Many initiatives that continue to engage our members today were seen as priorities:

Teacher Development: The disparities in the quality of mathematics education offered to learners of different races were too obvious to ignore. The majority of mathematics teachers had very limited training. Consequently teacher development initiatives were seen as an urgent priority. AMESA members in some areas embarked on a programme of workshops to enable educators to develop and improve their mathematics teaching abilities among themselves. Funding was sought to assist with teacher development in the different provinces. The first of the teacher development funders was Mastermaths.

Mathematics Competitions: Before the formation of AMESA, there were separate competitions run by the different associations. The Old Mutual National Mathematics Olympiad, although open to all, had very limited participation by black learners. In 1993, the AMESA challenges which began in the Western Cape began to spread to other provinces on account of the efforts of Alwyn Olivier and others.

Publications: Pythagoras, which started as journal under the Mathematics Association of South Africa (MASA) continued under the same name with Prof Michael de Villiers as editor. On average, 3 issues per year were published. AMESA News was published once a quarter.

Branch Activities: Branch activities began to spread slowly, with some branches being more active than others. The main activities were self empowerment workshops to improve content knowledge and teaching methodologies using local or invited expertise.

Relationship with National Department of Education. The ground work was done to ensure that AMESA was seen as the most important voice of Mathematics Education in SA. Every effort was made to ensure that AMESA was consulted on all matters relating to mathematics education in the country. The role of Professor Paul Laridon in these matters cannot be overstated. Thus AMESA began to play a significant role in the development and subsequent adjustments to the Outcomes Based Curriculum in mathematics.

The National Congresses. After the launch in Bloemfontein in 1993, the first national congress of AMESA was held at the University of Witwatersrand. As they do now, these congresses were organized to give our teachers exposure to mathematical thinking and practice within and outside the borders of our country.

THE KEY PRIORITIES FOR THE NEXT 5 TO10 YEARS

Teacher Development: Mathematics teachers appear to be in constant need of development. While any request for support should be viewed in positive light, it seems that the general desire is for more content training rather than updating on new trends and developments in the learning and teaching of mathematics. The reason for this is that mathematics is seen as a set of rules and procedures to produce answers to standard types of questions on a fixed set of topics. Mathematics teacher support should therefore focus on developing teachers as mathematical problem solvers. Having a problem solving attitude is certain to make teachers more confident to engage in the independent study of mathematical content.

Increase in Membership. For AMESA to engage in fulfilling its mission there has to be a radical increase in membership. The difficulty is that the majority of teachers who teach mathematics especially in the primary school do not see themselves as mathematics teachers. Most have no special interest in the subject and just go about teaching what they think they are expected to teach. Very few teachers of mathematics seem to display any confidence as mathematics teachers, and most of these few are members of AMESA already. If most mathematics teachers see themselves as being inadequate, then joining an organization like AMESA could be quite intimidating. Local AMESA leadership should attract the most successful mathematics teachers who may be viewed as a readily available resource in a locality.

It is very important that those involved in research in mathematics education are seen to be active in their own branches. Not many branches have the active involvement of lecturers from Higher Education.

Teacher Supply: The majority of the primary schools in our country do not have even one qualified mathematics educator, and a large number of high schools have qualified teachers with a limited capacity to solve mathematical problems. This makes the reading textbooks a major ordeal for teachers and creates a permanent dependence on workshops and other short term training. The department of education should be engaged on the issue of teacher supply. AMESA should play a role in determining the quantity and quality of mathematics teachers being trained.

AMESA's role in curriculum changes: While curriculum changes are always a necessary part of education policy, South Africa has had more than its share of changes for relatively short period of time. It is high time we reached some stability. Continual vacillating over what should and should not be included in the curriculum is a source of considerable unease among educators. Since most mathematics teachers have had limited training, every minor change creates a dependence on workshops and courses. A case in point is the exclusion/inclusion of Euclidean Geometry as a core part of the curriculum. We need to accept that mathematical content is only a means to enable our learners to develop into mathematical problem solvers. There will always be discontent among small groups of well meaning individuals; their lobbying should only lead to incremental changes. Without a degree of stability in the curriculum, innovation becomes rather difficult and research in mathematics education becomes an end in itself. I certainly do not believe that research in mathematics education should be an end in itself.

THE OPPORTUNITIES AND CONSTRAINTS IN AMESA'S STRUCTURES AND PRACTICES IN TERMS OF CONTRIBUTING TO THE DEVELOPMENT OF THE MATHEMATICS EDUCATION FIELD

Opportunities:

- We have the audience of the Department of Basic Education. We need to lobby to make the teaching of mathematics and science more attractive to school leavers. We need to partner with the department to ensure that mathematics teacher training is sufficiently rigorous to ensure that teachers entering the system are effective problem solvers. The department should be encouraged to provide incentives for mathematics and science teaches. The incentives could be monetary, and/or rapid mobility to higher posts for consistently good performance.

- Many of our branches are very active. These need to be further incentivized to attract more members by reaching out to those who feel marginalized from the community of mathematics teachers.

- Poor learner performance can certainly be viewed as an opportunity. There is no greater impetus that AMESA needs to drive research in mathematics education than the state of mathematics learning. Systemic evaluation in numeracy, the TIMMS study and the matriculation examination all point the need for more research.

Constraints:

- As noted already AMESA's relatively low membership is a major impediment to development. In the latter half of 90's AMESA membership rarely exceeded 1000, except immediately after a national conference. I am informed that the current membership stand at about 2000. This is extremely small for an organization that is vested with a huge responsibility. It is time that AMESA was adequately staffed with personnel to support more outreach and marketing work as well as further research and publishing.

AMESA'S ROLE IN RELATION TO SUPPORTING A PROFESSION THAT WORKS IN RESEARCH/EVIDENCE-INFORMED WAYS

I shall refer to a teacher development programme initiated by the MEC for Education in KwaZulu Natal. Grade 12 mathematics educators from schools that obtained a pass rate of less than 60% for mathematics were invited to capacity building workshops over weekends. There have been three such workshops held for mathematics thus far. The first was held from 18-10 February 2011, the second from 11-13 March 2011 and the third from 13 to 16 May 2011.

The educators reported to the venue at 16h00 on the Friday. They were required to write what is referred to as a pre-test. Thereafter they were work shopped on topics in the Grade 11 and 12 curriculum for the rest of the weekend until Sunday when they wrote what is referred to as a post-test. The pre-test and post-tests had a very similar questions.

Lead educators from better performing schools together with subject advisors were facilitators the workshops. The course entailed the solving of problems from past examination papers together with the associated teaching methodology.

Below is a summary of the statistics for the results of the pre and post tests for educators in one particular district for the first two of the sessions of training:

First Training Session

Statistical Data

	No Wrote	0-29%	30-39%	40-49%	50-59%	60-69%	70-79%	80-89%	90-100%
Pre-Test	49	9	10	8	13	5	2	1	1
Post-Test	49	4	6	4	5	12	5	8	9

	Total Wrote	Lowest %	1st Quartile	Median	3rd Quartile	Highest %	Average %
Pre-Test	49	12	34	46	54	94	45
Post-Test	49	16	44	66	80	98	62

Second Training Session

Statistical Data

	Total	0-29%	30-39%	40-49%	50-59%	60-69%	70-79%	80-89%	90-100%
Pre-Test	**59**	**5**	**13**	**8**	**13**	**12**	**6**	**2**	**0**
Post-Test	**59**	**2**	**0**	**4**	**10**	**14**	**12**	**10**	**7**

	Total	Lowest	1st Quartile	Median	3rd Quartile	Highest	Average
Pre-Test	59	10	38	54	62	80	50
Post-Test	59	24	58	68	80	100	69

A few simple observations

- 10 more educators joined the group in the second session.
- There was an improvement in the performance of educators from the pre-test to the post-test in both sessions
- The top 15 scores in the pre-test ranged between 62 and 80% while the top 15 scores in the post-test ranged between 80 and 100%.
- The bottom 15 scores in the pre-test ranged between 10 and 38% while the bottom 15 scores in the post-test ranged between 24 and 58%.

Some important research questions

All the teachers came from schools that failed to pass 60% of the learners in Mathematics in the 2010 National Senior Certificate examinations (the pass % used is 30%). Only 3 of the educators who attended the course are AMESA members. Five of the six facilitators used are AMESA members.

1 Three quarters of the educators could score no more than 55% in the pre-tests. What sustainable interventions can be made to bring these educators up to speed?

2 About 25% of the educators show that they have a reasonable to good capacity to solve grade 12 problems. What could be possible reasons for his/her inability to ensure a better learner success rate.

3 How can AMESA engage educators from poor performing schools in action research to encourage them to evaluate their practice?

4 Does AMESA attract only well performing educators?

CONCLUSION

There are about 26000 ordinary schools in South Africa. AMESA's challenge is to ensure that at least one mathematics educator in each of half the number of schools is recruited as a member. A membership of about 13000 is a target that AMESA should be working towards in order to make a significant impact on the Mathematics Education in our country.

LONG PAPERS

GROWTH OF STUDENTS' UNDERSTANDING OF PART-WHOLE SUB-CONSTRUCT OF RATIONAL NUMBER ON THE LAYERS OF PIRIE-KIEREN THEORY

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This paper discusses the growth of students' understanding of part-whole sub-construct of rational number on the layers of Pirie-Kieren theory. The design of the study was observational and employed a teaching experiment methodology. Students were observed over a number of teaching episodes on individual basis, as they worked on part-whole tasks in the classroom. The observations were video taped and transcribed. The result of the analysis of the observations (mathematical actions of the students) revealed that students' understanding of the concept of part-whole sub-construct of rational number was observed to grow along the layers of Pirie- Kieren theory. The study also identified significant instances of folding back as an extended element of the theory. It was concluded that the notion of folding back played a significant role in the growth of mathematical understanding.

INTRODUCTION

The Pirie-Kieren theory for the growth of mathematical understanding, and its associated model, is a well established and recognized theoretical viewpoint on the nature of mathematical understanding (see Kieren, Pirie, & Gordon Calvert, 1999; Pirie & Kieren, 1994). The significant part of the model is that, it adopts the approach that the processes of understanding are whole, dynamic, recursive, and non-linear. Building their theory from constructivist views of learning, Pirie and Kieren (1994) elaborate the constructivist idea of an individual understanding as the continual process of organizing and reorganizing knowledge structures (Von Glasersfeld, 1987). The theory offers a language for, and way of observing, the dynamic growth of mathematical understanding, and it contains eight potential levels for understanding. These are Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising (see Fig. 1)

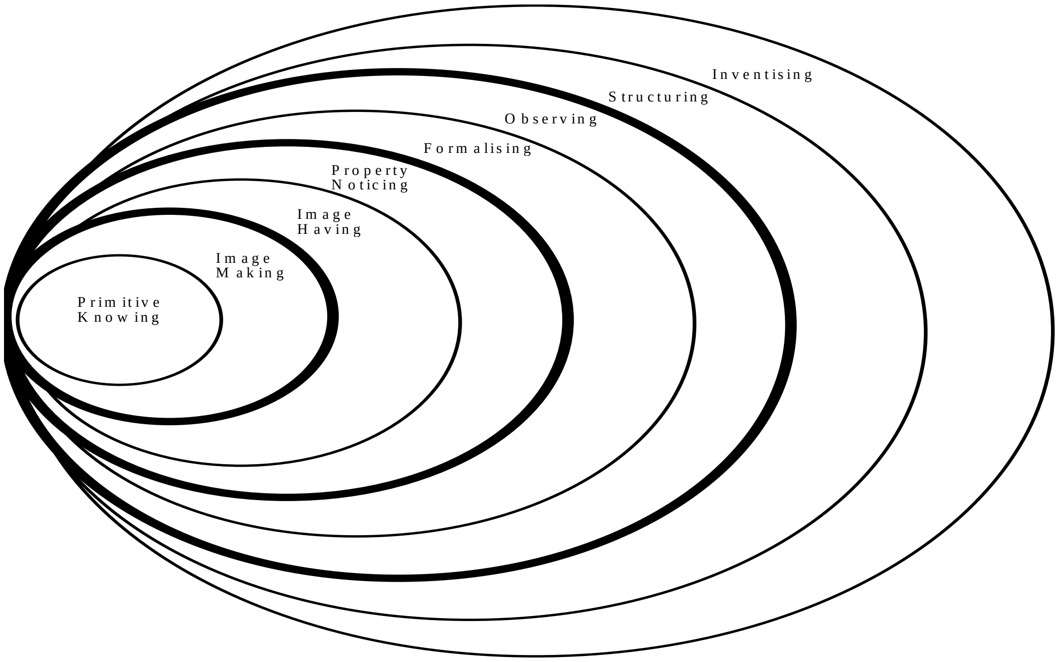

Fig. 1: The Pirie-Kieren model for the dynamical growth of mathematical understanding (Pirie and Kieren, 1994)

Brief definitions of the various layers of understanding actions are given here (see Pirie & Kieren, 1994, for full version). In addition, as the extracts of data drawn on in this paper do not include any cases of Observing, Structuring or Inventising, these layers are not considered here.

In the Pirie–Kieren Theory, *Primitive Knowing* refers to the knowledge that an individual brings to a setting. The process of growth of understanding begins at this level and it contains what a researcher assumes a learner can do at the beginning of instruction. *Image Making* refers to the level where a learner can make distinctions in his or her previous knowing and can use that knowledge in new ways that involve actions and activities with that knowledge. At the *Image Having* layer, the learner is no longer tied to an activity, he or she is now able to carry a mental plan for these activities with them and use it accordingly. *Property Noticing* it is a level of understanding which occurs when one can manipulate or combine aspects of his/her images to construct context specific, relevant properties. At the *formalising* layer, the learner is able to think consciously about the generalized properties and work with the concept as a formal object, without specific reference to a particular action or image.

However, a path of growing understanding for a learner working on a particular topic is not necessarily a mono directional one, outwards, through these layers. Indeed the theory states that when faced with a problem at any level that is not immediately solvable, an individual will need to return to an inner layer of understanding. This shift to working at an inner layer of understanding actions is termed *folding back* and enables the learner to make use of current outer layer knowing to inform inner understanding acts, which in turn enable further outer layer understanding. The result of this folding back is that the individuals are able to extend their current inadequate and incomplete understanding by reflecting on and then reorganizing their earlier constructs for the concept, leading to what may be termed a 'thicker' understanding for the concept.

The theory seemed to offer a means of investigating the growth of students' understanding of the concept of rational number. The concept of rational number in all its manifestation, has consistently given problem for primary and junior secondary school students. Many reasons have been given for this difficulty. The studies of Kieren over a period of 20 years and the Rational Number Learning Group of the University of Georgia in Atlanta have aptly demonstrated the difficulty (see Bell, Swan and Taylor, 1981; Kieren, 1980). The part-whole sub-construct, commonly expressed as a fraction, is often considered the basis of rational number knowledge and fundamental to the other interpretations (Ni and Zhou, 2005). The part-whole sub-construct is based on students' ability to partition a continuous amount or a set of discrete items into equal sized groups.

Against this background, this study aimed at applying the theory in order to analyse the mathematical actions of Nigerian junior secondary school II students on the development of the concept of part-whole sub-construct. Specifically, through the use of mapping, this is a technique in Pirie-Kieren theory that maps out the pathway of growth of mathematical understanding, indicating the ways in which the understanding actions of the learners shift within the layers of the theory.

METHODOLOGY

The design of the study was observational and employed the teaching experiment methodology. The teaching experiment according to Steffe (1983) involves a sequence of teaching episodes. A teaching episode includes a teaching agent, one or more students, a witness of teaching episodes and a method of recording what transpires during the episode. Two students Sabriyya and Nabil (pseudonyms) were involved in the teaching experiment. The students were observed over a number of teaching episodes on an individual basis, as they worked on part-whole tasks in the classroom. The observations were video taped and transcribed. Data collected was analyzed through an iterative process of viewing and re-viewing the video data while coding the mathematical actions of the learners. Supporting evidence such as observational notes taken by the witness of the teaching episode and copies of students' work on classroom mathematics tasks were also used for the analysis and a storyline was constructed. From the storyline a pathway of students' growth of understanding of the part-whole sub-construct of rational number was map out on the Pirie-Kieren model.

THE RESULT OF TEACHING EXPERIMENT WITH SABRIYYA

Sabriyya, a JSS II student was 13 years 10 months old and the session of the teaching episode with her lasted for 30 minutes 42 seconds. Prior to the interaction described below, Sabriyya had been introduced in the previous school year to the term "fraction". She exhibited the conception of the meaning of fraction as part-whole, given ½, $^4/_8$ and ¼ as an example of fraction. When tested, Sabriyya demonstrated that she could physically make models of fraction from the real life situations. Also, she could name and rename fractional parts and distinguished between numerator and denominator. With the help of the activity of paper folding;

She was given a task to perform successive halving. By first folding a paper into two equal parts; again and again…

Sabriyya engaged in such an activity. She very quickly internalized the activity of successive halving; she generated the members of the halving family as in (i) below

$$^1/_2,\quad ^1/_4,\quad ^1/_8,\quad ^1/_{16},\ \ldots \qquad \text{(i)}$$

When she folded the paper for the third time, she wrote $^1/_8$ without counting the number of divisions. When tested, she said "I just multiply by 2, because each part is divided into two pieces". She was asked to state which part is greater between ¼ and $^1/_8$. With confidence, she was able to demonstrate that ¼ is greater than $^1/_8$ by the use of paper folding. She further confirmed that the more the paper is folded, the more the size of the fraction reduces.

Similarly, another task was given to her to perform successive thirding by first folding a paper into three equal parts, again and again…

With discomfiture, Sabriyya said "into three equal parts!!!" However, as she engaged in the activity with the help of the teacher, she was able to fold the paper into 3 equal parts. She was asked to predict the number of divisions when the paper is folded for the third time into another three equal parts. She incorrectly stated that 12 divisions was the answer. With the intervention of the teacher, she recalled the case of successive halving and said “No sir, the answer is 27 because 9x3 = 27, since the paper is folded into another 3 equal parts”. She generated the members of the thirding family as follows;

$^1/_3$, $^1/_9$, $^1/_{27}$, etc ... (ii)

Having done the two practical activities, Sabriyya was challenged to generate the members of the fifthing family without performing any practical activity. She wrote

$^1/_5$, $^1/_{25}$, $^1/_{125}$, etc ... (iii)

When tested, she said “I got this numbers by calculating 5, 5x5 and 25x5 etc, because the paper is to be divided into five pieces. So I just multiplied by 5”. She states two observations: (i) the size of the fraction decreases as you continue folding the paper and (2) one will generate the members of the set in successive partitioning by either counting the number of divisions after folding the paper or by multiplying with a common factor.

Sabriyya was given a model of fraction circles sets into different segments and shaded with colours; black, yellow, brown, blue, pink, purple, green and red (see Figure 2)

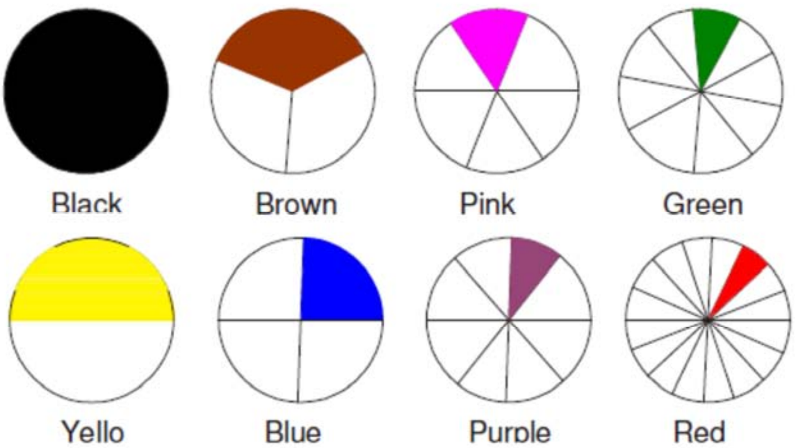

Fig. 2: Fraction Circles

She was informed that black is a whole. Quickly she internalized the idea contained in the fraction circle. With black as a whole, she was able to confidently name most of the other segments of the fraction circles i.e. yellow = ½, Brown = $^1/_3$, blue = ¼, pink = $^1/_6$, and incorrectly she said Green = $^1/_7$. When tested, Sabriyya claimed that green was divided into 7 places. With the help of the teacher; having counting, she discovered that green = 1/9.

When the unit was decomposed, having yellow (½ of the circle) as a whole, Sabriyya was asked to name the other parts in relation to the segment of yellow as a whole. It started by asking the value of black. She quickly said black is 2, because there are two yellows in one black. When asked for the value of blue, she said 'blue is ¼' (referring to the value of blue as in the complete circle is a whole). The teacher asked her to count the number of the segment of blue in yellow. She counted two segments of blue in yellow and quickly said the value of blue is ½. When tested, Sabriyya said because the equivalent of the portion of yellow was divided into two places, hence one part is ½. With ease, she was able to name the segments of pink, purple and red correctly in relation to the segments of yellow as a whole.

Similarly, the unit was decomposed having brown (1/3 of the circle) as a whole. Sabriyya was asked to find the value of 1 green in that situation. Very quickly, she said 1/3, because green was divided into 3 places in brown. When she was asked for the value of 3 pink in relation to brown (because the answer is greater than 1) she became confused. She said "3 pink is greater than one brown". When the teacher insisted for the numerical value of 3 pink, she said 3/1. When asked to explain, she sat in silent concentration. The problem was reduced, by asking her the value of only one pink. Sabriyya quickly said ½, and then the problem was restated for her. She internalized the problem, coming to understand that 3 pink would be ½ + ½ + ½ = 1½.

THE RESULT OF TEACHING EXPERIMENT WITH NABIL

Nabil, a JSS II student was 13 years 6 months old and the session of the teaching episode with him lasted for 28 minutes 4 seconds. Prior to the interaction described below, Nabil also had been introduced in the previous school year to the term "fraction". He demonstrated that he could physically make models of fraction from real life situations. The same three tasks given to Sabriyya were given to Nabil also.

In the task of successive halving, Nabil engaged in such an activity. He very quickly internalized the activity and generated the members of the halving family as in (i) above. When tested, he was asked to predict the number of divisions, when the paper is folded for the fourth time. He said it would be 16, because, before it was 8 and if you multiplied it by 2, it becomes 16 since the paper is folded twice. He also observed that in the set of the numbers he generated, the denominator has a common ratio of 2. He also discovered that the size of the fraction is decreasing as you continue folding the paper.

In the task of successive halving, it was very challenging to him. But at last without any help given by the teacher, he was able to fold the paper into 3 equal parts. For the next successive thirding, he accomplished with ease, as he became familiar with the activity. He generated the members of the thirding family as in equation (ii).

When tested, whether he conceptualized the activity of successive partitioning, he was asked to find the members of the fifthing family without carrying the activity of paper folding. With ease, he generated the members of the fifthing family as in equation (iii).

Nabil was asked to explain how he generated the numbers. He said the denominator was multiplied by 5, because the paper is to be folded in five equal parts. He came to understand that by successive partitioning, the number of partition is to be multiplied in each case to generate the members of the family.

In the third activity, Nabil was given the model of fraction circles as shown in Fig. 2 and was told that black is a whole. Quickly he internalized the idea contained in the fraction circle. With black as a whole, Nabil was able to confidently name all the other segments of the fraction circles i.e. yellow = ½, blue = ¼, Brown = $^{1}/_{3}$, pink = $^{1}/_{6}$, Purple = $^{1}/_{8}$, and Red = $^{1}/_{16}$. When tested, for Red = $^{1}/_{16}$, he said because there are 16 divisions and only one is marked for red.

When the unit was decomposed, having yellow (½ of the circle) as a whole, Nabil was asked to name the other parts of the fraction circle in relation to the segment of yellow. It started by asking the value of black. He said black is 2, because if you multiplied yellow by 2, you would get black. When asked for the value of blue (1/4 of the circle) and purple (1/8 of the circle), he said ½ and 1/3 respectively. When tested why purple was 1/3 he said because blue = ½, therefore purple would be 1/3. Nabil further pointed at the circles and he tried to convince the teacher by relating the segment of blue with that of purple. He said blue is half and purple is half of blue. The teacher asked whether ½ of ½ is 1/3. He quickly rejected his earlier answer and said purple is 1/4. When tested, Nabil was asked for the value of pink (1/6 of the circle) in the same situation as above, he instantly said, it is 1/3, because pink is less than blue and greater than purple. Therefore it should be 1/3.

Similarly, the unit is decomposed having brown (1/3 of the circle) as a whole. Nabil was asked to find the value of black in this situation. Very quickly, he said 3, because 3 brown would complete the circle. He was further asked for the blue of Pink (1/6 of the circle) and Green (1/9 of the circle). He said, Pink is ½ and Green is quarter, because green is half of Pink. When tested, he insisted that green was half of a pink. The teacher asked him to count the divisions in pink and green and compare them. He counted and got 6 and 9 partitions for pink and green respectively. He was further asked that, was 6 half of 9? With surprise, he said 6 is not half of 9. After quite sometime thinking and working on the model of the fraction circle, he then said green would be 1/5. The teacher asked him whether 5 of green is equivalent to one brown. With this question, he quickly changed and said it must be 1/3. The answer is correct, but he could not give clear reason for the value of green as 1/3.

ANALYSIS AND OBSERVATIONS

The process of coming to understand starts at a level called primitive knowing. According to the theory, primitive knowing does not imply low level mathematics, but rather the starting place for the growth of any particular mathematical understanding. For the growth of initial understanding of part-whole sub-construct, the teacher/researcher assumed that the students already had an idea of partitioning and distinction between "how many" (the numerator) and "how much" (the denominator). It was observed that, at least the two students had a practical knowledge of partitioning.

At the second level, the learner was asked to make distinctions in previous knowing and use it in new ways. In the narrative above, the students used their previous knowledge of partitioning to perform the activity of successive partitioning. It was the purpose of this activity (using paper folding) to occasion the students to generate the members of the succession and to record and reflect on those actions. This mode of understanding is called image making according to the theory.

It was observed that, the students can generate the members of the succession without having to perform the activity of successive partitioning using paper folding. For example, they were asked to generate the members of fifthing family without performing the activity of successive partitioning. Their initial understanding revealed that, they already have an image for successive partitioning. This has now been supplanted by a new image formed as a result of image making activity suggested by the teacher. This level of understanding is termed image having. At this level, a person can use a mental construct about a concept without having to do the particular activities which brought it. The students were freed from the need to perform the successive partitioning in order to generate the members of the succession.

The next level of understanding occurs when one can manipulate or combine aspects of ones images to construct context specific, relevant properties. In the model of the fraction circle, when the unit was decomposed, the students were found to use their image of part-whole which fit and their idea of equivalent of fractions to generate a means of getting a part in relation to the segment considered as a whole. This kind of understanding referred to property noticing.

At the fifth level of understanding, the learner is able to think consciously about the generalized properties and work with the concept as a formal object, without specific reference to a particular action or image. In the case of the students observed, they were found to state a number of segments equivalent portion of fraction greater than one. This level of understanding is termed formalising.

The growth of understanding of the two students reported in this study do not occurred in a linear manner. Both of them were seen to fold back during the growth of their understanding of part-whole sub-construct. For example, Sabriyya was seen to fold back from image having to image making level as shown in fig. 3. She was asked to predict the number of divisions, when a paper is folded for the third time by successive thirding. She gave an incorrect answer, as the interaction goes on; Sabriyya realized her mistake and recalled the previous episode. This is presented below:

Teacher:	Please could you predict the number of divisions, when the paper is folded for the third time by successive thirding?
Sabriyya:	it will be 12 divisions
Teacher:	How?
Sabriyya:	because 8 parts had been divided into 3
Teacher:	From 9 divisions or 8?
Sabriyya:	No sir! It is 27 because 9x3 = 27 since the paper is folded into another 3 equal parts as in the case of successive halving.

Similarly, another folding back occurred when Sabriyya was asked to find the value of blue (1/4 of the circle), having yellow (1/2 of the circle) as a whole. She did not put into consideration that the unit of the circle was decomposed. As presented below, she later came to understand her weakness…

Teacher:	Having yellow as a whole, what is the value of blue in relation to yellow?
Sabriyya:	Blue is ¼
Teacher:	Are you sure of your answer?
Sabriyya:	Do you mean the entire circle is yellow?
Teacher:	No, only the segment of yellow is a whole.
Sabriyya:	ah…! It must be ½, because the equivalent portion of yellow is divided into two places, hence one part is ½.

Nabil was also found to fold back from one layer to another. For example, he was asked to fold a paper into three equal parts. It was really challenging to him, but he returned to the primitive knowing. However, this returned gave him the power to perform.

Teacher:	Take a paper and fold it into three equal parts.
Nabil:	(…he folded the paper into parts that are not three and kept changing the paper). After a while, Nabil sat in silent concentration. He then proceeded and was able to perform the task as required.
Teacher:	Are you sure the parts are equal
Nabil:	(with confidence…) yes
Teacher:	Why did it take you a lot of trials before you got it right?
Nabil:	Yes, I did it for several times, but it was later I recalled the concept of partitioning into equal parts

Nabil was working at the image making layer and he clearly saw the need to use his previous knowing in the new task that was given to him to perform. With the returned to primitive knowing layer, enables him to make use of current outer layer knowing to inform inner understanding acts, which in turn enable him to fold the paper as required.

This finding was in agreement with the report of study conducted by Martin, LaCroix and Fownes (2005) that folding back occurs with a purpose, to extend one's existing understandings which have proved to be inadequate for handling a newly encountered problem. It is in response to an obstacle that the learner re-visits earlier understandings, aiming to modify, collect, or build new conceptions. This will allow the difficulty to be overcome through an extended understanding of the topic. Fig. 3

illustrates the pathway of Sabriyya and Nabil's growth of understanding. This is represented by a line, indicating the ways in which their understanding actions shift within the layers of the theory.

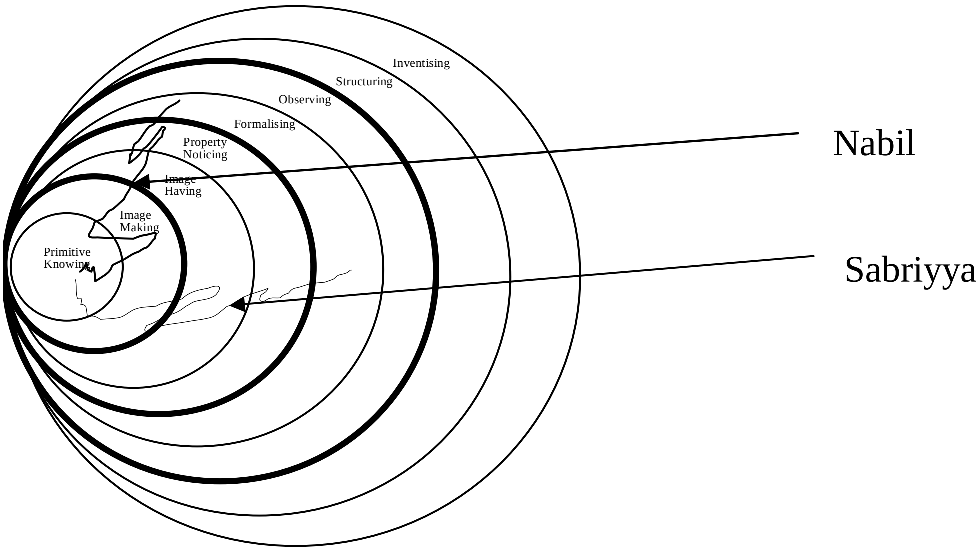

Fig 3: Pathway of Sabriyya and Nabil's growth of understanding of part-whole sub-construct on the layers of Pirie-Kieren theory

CONCLUSION

This study has been offered as a practical illustration of the application of Pirie-Kieren theory, specifically on the way of getting at the complexity of growing mathematical understanding. From the findings of this study, it was concluded that folding back is a factor to the students' growth of understanding of part-whole sub-construct of rational number. The subsequent study of this aspect of the theory revealed a greater insight into the process through which this growth occurred. This illustrated the power of the Pirie-Kieren theory as a theory for, not a theory of, the growth of mathematical understanding, and as such it is validated by its usefulness to someone seeking to make sense of the growing mathematical understanding of learners.

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DESCRIBING AND ANALYSING GRADE 10 LEARNERS' DESCRIPTIONS OF THE SYNTACTIC RESOURCES THEY USE TO TRANSFORM EXPRESSIONS

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In this paper I present a description and analysis of the syntactic resources that top Grade 10 learners from three different schools use to transform mathematical expressions. The most successful Grade 10 learners from the schools were asked to explain the reasoning they use in producing their solutions to selected examination questions on their 2010 mid-year mathematics examinations. Their descriptions of their operational activity reveal that, although they are able to competently construct correct solutions to the selected questions, some of the syntactic resources they use appear to be problematic. Certain sequences of operations are unnecessarily complicated, suggesting that the algorithms learnt at school can have an effect on the way learners reason mathematically.

INTRODUCTION

In this paper I describe and analyse the computational resources used by three Grade 10 learners in their solutions to mathematics problems on their 2010 mid-year examinations. These learners were participants in a larger study concerned with describing the mathematical activity of twenty Grade 10 learners from three different schools, judged to be the most mathematically competent learners in their respective classes by their teachers, and who were also the most successful learners according to their performances in their 2010 mid-year examinations. The purpose of the exploration of the mathematical activity of the three learners is to develop adequate descriptive and analytic resources for producing data on what learners constitute as mathematics and how they do so, in terms of the mathematical operations that they employ.

In this research project we are particularly interested in what the most competent learners constitute as mathematics because, we assume, that what such learners produce is a reasonable proxy for what is taught as mathematics by their teachers. Our assumption is based on a couple of propositions. The first, derived from Bernstein (1996), states that pedagogy is necessarily evaluative and that pedagogic evaluation makes available to the learner criteria for the recognition and realisation of what is expected by the teacher. The second proposition informing our assumption is one that Davis (2011: 100) explains:

> *It is not unusual to find alternate operations, or even pseudo-operations, replacing the*

operations indicated by mathematical statements in the pedagogic situations of schooling. However, it is not always the case that the manipulations introduced by teachers and/or their learners are operations in a mathematical sense because the structure of a manipulation would have to conform to that of a function to be considered an operation.

Given that it is possible to replace an operation or operations by another or other operations and thereby use different rules to achieve the same outcome, then it is probable that teachers select certain formulations that they feel are the easiest for their learners to apply in the particular mathematical context. What are these resources that are taught in schools and do they achieve what the pedagogy intends? Do they provide the learner with the recognition and realisation rules necessary for what Ma (1999) refers to as "a profound understanding of fundamental mathematics"?

LITERATURE REVIEW

Since Skemp (1971) and the contributions of the concepts of 'relational' and 'instrumental' understanding to that work from Mellin-Olsen's work, published in Skemp (1976), there have been attempts to describe the nature of the procedures that learners engage with when studying mathematics. Hiebert & Lefevre (1986) describe conceptual and procedural learning in the form of an opposition with procedural learning often considered as something not good. Star (2005: 405) claims that "these distinctions are limiting" and are a reason for the relative lack of research on procedural knowledge. He refers to ***deep*** as opposed to ***superficial*** procedural activity and makes a call for additional research on the character of the knowledge that supports learners' ability to perform procedures (ibid.).

Skemp (1976: 21) describes resources like "'borrowing' in subtraction, 'turn it upside down and multiply' for division by a fraction, 'take it over to the other side and change the sign'" as common in learners' work. Tall et al (2001: 15), in their description of learners' procedures, say that it is most likely that "[t]he symbols have little meaning other than carrying out learned rules impressed on them by their teacher". They are talking about resources like "'change sides, change signs', 'move the numbers over to the right', 'move the *x*s to the left', 'divide both sides by the coefficient of *x*'". Lima & Tall (2010: 17) argue that

[t]hose who develop a flexible proceptual[4] knowledge structure in arithmetic have a powerful generative engine to derive new facts from known facts while those who operate in a procedural manner have longer sequences of operations to perform that make arithmetic even more difficult for those who are already struggling.

In research that is perhaps most closely aligned to the work discussed here, Filloy,

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[4] Tall uses *procept* to refer to the flexible movement between the process and concept necessary for mathematical proficiency (Gray & Tall, 1994).

Puig & Rojano (2008: 115) make an “analysis of certain general patterns that were observed in the performance of children in the upper stratum …” and describe a range of resources that learners use to transform expressions, from the syntactic or operational to the semantic. Davis (2010a; 2010b) points to the likelihood that some of these syntactic resources in use at school complicate or make unintelligible the mathematics concepts being taught.

Skemp (1971: 118; 1976: 21), both in his early work and his descriptions of ***instrumental*** and ***relational understanding***, referred to teaching learners “rules without reason” as opposed to teaching for understanding. In his initial description of the instrumental-relational understanding opposition he quotes a 1960 textbook that says: “we use the rule that when we change the side we change the sign” (Skemp, 1971: 119). In response Skemp (1971: 119) argues that if

> *all that is wanted is to be able to solve equations of this kind quickly and efficiently, such a method is adequate. If, however, any importance is attached to understanding what one is doing, then it is not. And this understanding is not just a luxury which makes the task more pleasant; it is a necessity if one is to be able to adapt one’s knowledge to new situations.*

These “rules without reason” are a shibboleth in mathematics education research. Why are these rules “without reason”? If a learner can apply a rule and obtain a successful outcome surely the mathematics produced has reason. Can we then cry: “you have no understanding”? What seems more meaningful is that we are able to describe the learner’s operational activity in a sufficiently robust manner so that we are able to say something about the syntactic resources being used.

METHODOLOGY

We draw on the theoretical resources outlined by Davis (2010c; 2011), with particular emphasis on the fact that the operations that populate mathematics are functions and functions have unique outputs for given inputs. Davis (2011: 98) refers to the fact “that all processes that are to be deployed as operations in mathematical work in pedagogic situations are functions.” We use this to develop our description of the syntactic resources used by learners. The important realization is that it is possible to replace an operation or operations by another or other operations and thereby use different rules whilst preserving a function as a relation between a set of inputs and a set of outputs. By observing the rules that are mapped out by the operations that learners use, we expect to find a reasonable proxy for what is taught as mathematics by their teachers. Our assumption is that learners’ mathematical activity is structured by the evaluations of their teachers.

The empirical setting we constructed is one where the most competent learners are interviewed about a solution that they produced, correctly according to their teachers, in their midyear exam. With a selected exam question as the focus of the interview, the learners were asked to explain their reasoning. The interviews were recorded on video and learners were encouraged to write down any of their calculations.

We can describe the schools that these learners attend as follows:

A. a top private school
B. a private school for disadvantaged learners
C. an ex-DET school

For the purposes of this report we make use of descriptions of the reasoning of only one learner from each of the above three types of school. Each learner's explanation of the syntactic resources that they use is scrutinized with a mathematical lens. The transcripts of the descriptions, as well as any additional notes that the learners make, are the primary archive from which we generate data for descriptions of the mathematics constituted in the selected pedagogic situations, and we do this by describing the operations and operation-like manipulations that are encountered. It is often helpful, although not presented in this report, for a concise mathematical description to present the objects and operations as sets and then to decide whether the elements of these sets are compossible. An example of a set theoretic description might be $(\mathbb{Z}, +)$, where integer addition is described with the objects being integers and the operation addition.

We take an extensional approach where we describe and analyse whatever emerges from the learners' descriptions, attempting to avoid any judgment or any intention to extract a particular selection of data. The steps of the learners' routines and the operations that constitute these steps for transforming the expression; namely the exam question, are listed so as to get at the nature of the functions used. The three learners' descriptions that we select for this report are chosen mainly because the problems that they engage with are similar in that they involve finding the solution of equations. Our method quite easily extends to a description of the syntactical resources utilized in the transformation of any mathematical expression. The intention is to get at a description of the particular routine that a learner presents for evaluation and then to try to get at what is taught as mathematics by their teachers, namely what these top learners acquire from the pedagogy?

ANALYSIS

We describe and analyse interviews with the following learners (pseudonyms used).

Laura's description (School A)

We might describe the steps in Laura's solution to the problem where she is asked to solve for *x* in $2x^2 + 5x = 12$ (Figure 1), with the help of her explanation extracted from the interview as follows (Table 1).

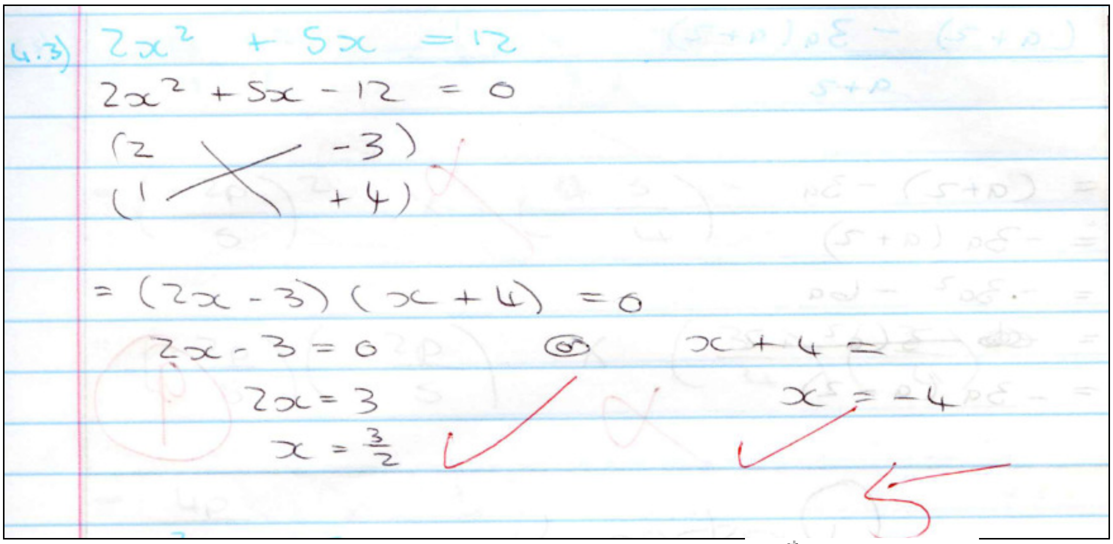

Figure 1: Laura solves for *x*: $2x^2 + 5x = 12$

STEP	Description	Laura's comment
1	assess the spatial distribution of symbols	"you can't solve it like this, if the twelve is here"
2	'change sides, change signs'	
3	spatial observation	"now there is nothing left there"
4	counting , ordering and classification of terms	"you must identify it as a trinomial"
5	quadratic factorisation (version of 'cross-method') involving a sequence of operations	
6	'change sides, change signs'	
7	Division $(\mathbb{Z}, \div)$ followed by repeat of STEP 4	

Table 1: The steps in Laura's calculation

Laura's first step (Table 1, STEP 1) is, of course, not mathematically speaking true, as she might realise when she sees a solution that entails completing the square. She could also write $x(2x + 5) = 12$ and extract solutions from the possible rational divisors of 12, but then that would miss the point and that is to recognise the quadratic form and its graphical representation. Laura limits the way that she thinks about the equation and she focuses on an algorithm that will reproduce what she has learnt. She uses an almost standard 'change sides, change signs' resource (Table 1, STEPS 1&6) to perform the operation of 'what you do to one side you do to the other'. It appears, however, that her description involves what has been referred to as 'domain shifting' (Basbozkurt, 2010; Davis, 2010a), in fact two shifts, where an integer is seen as a character string and then again as an integer. She says: "If you bring it over it becomes a minus".

Another example of a somewhat complicated function is her resource for dealing with the signs of the terms in her version of what is often referred to as the 'cross-method' (Table 1, STEP 5). She says, rehearsing her algorithm when referring to the expression $2x^2 + 5x - 12 = 0$, that "if it is a plus then the signs are the same", meaning that by observing a selected sign in the quadratic equation then in $x \pm a = 0$ or $x \pm b = 0$ in $(x \pm a)(x \pm b) = 0$ we can choose the signs correctly. We observe a dominance of using how things look as a resource as she makes spatial observations (Table 1, STEPS 1&3) like: "Now there is nothing left there". A discussion of the spatial distribution of symbols is more fully described by Johnson & Davis (2010). Laura has a tendency to engage in unnecessary routines as seen in her 'count terms', 'order' and 'classify' routines (Table 1, STEP 4) for deciding if what is present is a trinomial when what she needs to decide is whether to utilize syntactic resources associated with solving a quadratic equation. Such a routine appears potentially problematic because quadratic equations can also have less than three terms and so not be recognised as trinomials. These instances, notwithstanding, she is successful in using the syntactic resources that she has acquired from her schooling in mathematics, like addition, subtraction, multiplication, division and factors. Although she arrives at an ultimately correct transformation of the expression engaged with, she verbally describes her solution as $x = 3/2$ AND $x = -4$ rather than $x = 3/2$ OR $x = -4$.

Ann's solution (School B)

Anne is asked to solve for *x* in $\frac{2x}{3} - 3 = 4(x - 1)$.

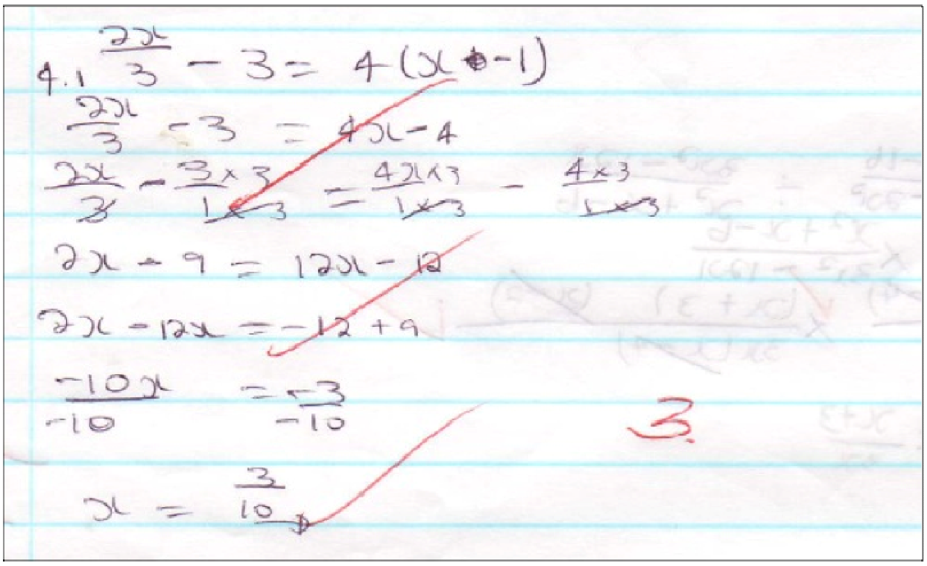

Figure 2: Ann's solution to "Solve for *x*: $\frac{2x}{3} - 3 = 4(x - 1)$"

The steps in Ann's algorithm (Figure 2) are described in Table 2. Ann appears to make the correct use of the mathematical rule for distribution in STEP 1, although she refers to it as "multiply". From STEP 2 to STEP 5 (Table 3) the expression

$\frac{2x}{3} - 3 = 4x - 4$ is transformed into the expression $2x - 9 = 12x - 12$. As described in this extract we see how Ann successfully uses a lengthy routine to achieve a simple operation of multiplying by 3. Her decision to engage with fractions and to 'cancel the denominators' is certainly successful, but she appears to closely regulate what she does by remembering what she has been taught to do when faced with an expression like $\frac{2x}{3} - 3 = 4x - 4$. Why would a top learner make such a move if it were not to satisfy the evaluative criteria?

STEP	**Description**
1	Algebraic multiplication
2 2.1 2.2	Finding LCD using a sequence of operations Identify fraction with + or – signs List denominators: 3;1;1;1;1 LCD: $3 \times 1 = 3$
3	Multiply both sides by $\frac{3}{3}$
4	Cancel all denominators
5	Multiply out the numerators
6	'changing sides, changing signs'
7	Subtract 'like' terms
8	Divide both sides by -10

Table 2: The steps in Ann's algorithm

TRANSCRIPT	**NOTES**
and therefore	->STEP 2 (finding LCD)
I looked for the denominator, the lowest common denominator which was three	$\frac{2x}{3}$ 3 $4x$ 4 LCD:3
{ I wanted to find the common denominator because I have got operation signs which are negative, they are both negative … therefore I had to find for a common denominator so that I can add or maybe subtract these numbers, these like terms. If they were maybe three, two, four and five I wouldn't be able to subtract them because they don't have the same denominators.}	To SUBTRACT like terms that are fractions requires making denominator the same
because	Regulation from fraction

	algorithm
if each and everyone here has a denominator of one	$\frac{2x}{3}$ $\frac{3}{1}$ $\frac{4x}{1}$ $\frac{4}{1}$
therefore	
I have to make it three	Regulation from addition of fractions
and	->STEP 3
I did … I multiplied all the other sides	$\frac{2x}{3}$ $\frac{3\times3}{1\times3}$ $\frac{4x\times3}{1\times3}$ $\frac{4\times3}{1\times3}$
after that	->STEP 4
I cancelled all the denominators	$\frac{2x}{\cancel{3}}$ $\frac{3\times3}{\cancel{1\times3}}$ $\frac{4x\times3}{\cancel{1\times3}}$ $\frac{4\times3}{\cancel{1\times3}}$
	->STEP 5
	$2x$ 9 $12x$ 12

Table 3: Extract (Table 2, STEPS 2-5) from Ann’s description of her reasoning

The syntactic resources indicated in Anne’s solution suggest an unnecessarily overcomplicated approach to transforming the expression. We might also question the reason that Ann’s method is acceptable to her. When viewed from a mathematical stance her algorithm lacks functional simplicity; after all, why not just multiply by 3? Also, when making the denominators the same, she refers to ‘cancelling’ and does not appear to be multiplying by 3. Although, we can with some effort, construct a function which allows the ‘cancelling of denominators’, it is not at all clear what type of operation we are using when the syntactic resource is ‘cancel’. Anne also uses spatial organization and sees an expression as a collection of symbols and sometimes appears to treat numbers as if they are part of a character string.

Ann appears aware of the need to preserve equality in the equation and yet decides to use the algorithm she has learnt, albeit unnecessarily. When she transforms $-10x = -3$ (Table 2, STEP 8) into the solution $x = 3/10$ she says: “I divided by the denominator … I mean by the coefficient of negative ten … both sides.

Here we see her description embedded with what Freud (1901) called *parapraxes*, described as slips that people make where unconscious motivations are revealed, namely that she understands what is required in the transformation of an equation but uses the algorithm she has learnt regardless of this knowledge. It is likely that other learners might simply focus on the distribution of symbols and ‘how things look’ in attempting to reach a solution. For the interest of the reader who might not consider it possible to transform this expression without resorting to complicated manipulations, we describe an algorithm (see Figure 3) that contains syntactic resources referred to in Gripper (2011), designed to avoid the problematic manipulations described earlier.

$$\frac{2x}{3} - 3 = 4(x - 1)$$

$$\therefore \frac{2x}{3} - 3 = 4x - 4$$

$$\therefore 2x - 9 = 12x - 12$$

$$\therefore 12x - 10x - 9 = 12x - 10\left(\frac{3}{10}\right) - 9$$

$$\therefore x = \frac{3}{10}$$

Figure 3: Using alternative resources to solve for *x*

Jacob's description (School C)

Solving for *x* and *y* in simultaneous equations where $x + y = 7$ and $2x - y = 11$.

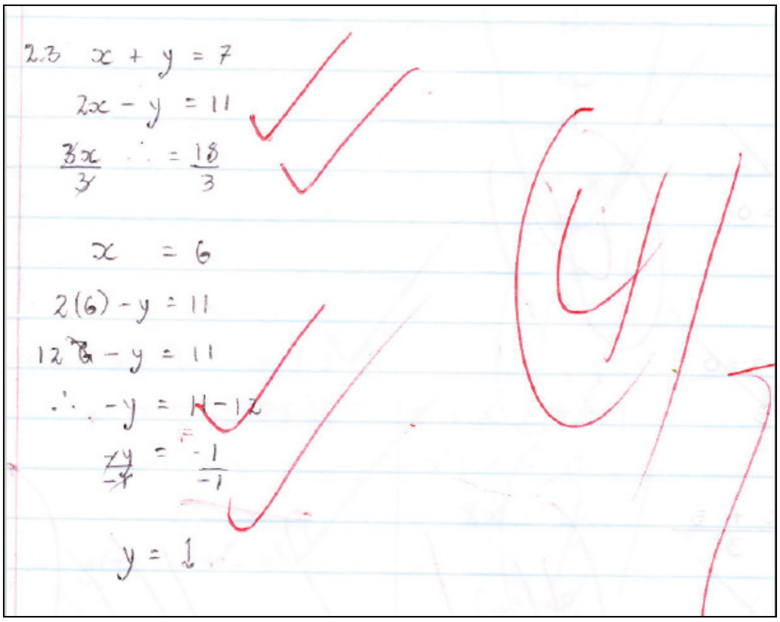

Figure 4: Jacob's solution to simultaneous equations

STEP	Description	Jacob's comment
1	Addition of expressions	"if I subtracted I get here … there's no way that I can find *x*"
2	Find *x* by adding and dividing or 'cancelling'	
3	Find *y* by substitution, multiplication, 'change sides, change signs', subtraction and division	

Table 4: The steps in Jacob's algorithm for solving simultaneous equations

Jacob correctly solves a simultaneous equation where $x + y = 7$ and $2x - y = 11$ (Figure 4). What is interesting is that his initial transformation (Table 4, STEP 1) that adds the two expressions most likely comes from an algorithm that he has learnt in an initial encounter with simultaneous equations. Such an observation is suggested by the method used by the teacher who constructed the memo (not shown). Her algorithm that includes a substitution, is ultimately more useful as it can be applied to simultaneous equations constituted by expressions of a higher order. Jacob's 'addition method' will lose its generalizing power as he attempts to solve other simultaneous equations that he will encounter at school. Although he might manage to change his method, it is of course likely that other learners might not manage such an adjustment. Jacob also uses the 'cancelling' technique (Table 4, STEP 2), as well as 'change side, change sign' (Table 4, STEP 3). He also has the tendency to reject syntactic resources that do not echo the algorithm that he has learnt. He says: "There's no way that I can find *x*." He rejects an algorithm for subtracting and chooses adding whereas mathematically both these operations can be described as having the same outcome in the context of the problem. He could, of course, first multiply by -1 before subtracting to achieve the same result as adding. In this regard Jacob has the tendency to limit the mathematical possibilities in the situation.

Summary of results

In summarising the operational activity in these instances, we see that it is sometimes the case that the learners make use of manipulations like 'cancelling' or 'change sides, change signs' that are not directly aligned with the mathematical operations they do the work of. This appears to be a common, almost universal resource, despite Skemp's (1971) earlier warning to teachers that the way such a resource is taught and learnt can be problematic. It is also possible that when the objects that learners operate on are noted (as in 'change side, change signs'), there is the occasional domain shift, where numbers are transformed into character strings. This can appear as magic. In general, it appears as if the operatory space described by these learners is somewhat complicated and at times unnecessarily so, especially given the exam conditions and the few marks allocated to the questions. There is often the tendency to rehearse techniques that should be second nature by Grade 10. The notion that is mostly regulative is the learnt algorithm itself, rather than mathematical thought.

DISCUSSION

We take the view that the realisation of an intelligent procedure is a goal of mathematics. In doing this we distance ourselves from the debate around the conceptual/procedural route to gaining an understanding of mathematical ideas in pedagogic contexts. Such notions, expressed in papers like Hiebert & Lefevre's (1986) seminal work on the procedural-conceptual opposition in mathematics education and the work of researchers we have referred to, like Tall (2009) who sets the acquisition of procedure beneath the axiomatic in the hierarchy of his ‘three worlds of mathematics’.

Instead we take a lead from Badiou (2008: 101), when discussing the relation between philosophy and mathematics, reminds us that: “Proceeding philosophically, Plato established mathematics at the frontier between thought and the freedom of thought”. This is because mathematics gives us the possibility of escaping from opinion, albeit within the constraints of available intelligible thought. Mathematics is, according to Plato (Badiou, 2008), a condition of thinking; it is a procedure that is existent and available for this purpose. The aim, according to our thinking, should be to acquire a procedure that can silence all criticism. The important point that Badiou (2008: 103) makes is that one is “‘forced’ to proceed according to the intelligible”. In order to silence criticism it is therefore necessary to develop mathematical procedure intelligently and succinctly. In the words of Badiou (2006: 120): “Mathematics is accomplished by axiomatic decisions that set up a possible universe as real. The outcome incurs logical constraints.” It is these constraints that describe the frontier between thought and the freedom of thought, emerging as relevant procedure. The axioms are the building blocks of any procedure.

In the light of such, it appears that learners are not encouraged by the pedagogy to select and refine the syntactic resources that they use. The pedagogy very often does not appear to raise the acquisition of an incontestable procedure to the zenith of the expression of mathematical thought. A situation where the pedagogy encourages learners to use the algorithms that are presented as the method expected by the teacher, is one where the learners are not encouraged to think mathematically. Making use of the way things look has a tendency to undermine the propositional nature of mathematics, weakening the chance that learners have to generalize their techniques. For those learners who are more capable of applying their learnt procedures in different situations, this might not at first glance appear as a problem. However, the use of some of the syntactical resources we have described could certainly prove a problem for the majority of learners. For a discussion about the grounding of mathematical activity as resources for describing the regulation of learner’s activity in the pedagogic situations of schooling, we refer the reader to Davis (2010d).

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THE GAP BETWEEN THE IMPLEMENTED AND INTENDED GRADE 10 TO 12 MATHEMATICS CURRICULUM

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This study investigates how Grade 10 to 12 teachers in Gauteng implement the National Curriculum Statement (NCS) in their classrooms. The assumption was that the best learner's workbook, as selected by the teacher, should reflect the content coverage, time spent on a topic, as well as the depth of curriculum implementation. Learners' workbooks were therefore analysed in terms of time spent on topic, curriculum coverage and the cognitive demand of activities. According to the workbooks from the sample schools, the average number of active learning days was 48.5, 56.6 and 53.9 for the entire school year for grades 10, 11 and 12 respectively. The results show that the main focus of workbook activities is the development of knowledge and routine procedures. Very little work is done on the higher levels of cognitive demand. In fact, only the three top performing schools did activities that focused on the development of critical thinking and problem solving.

INTRODUCTION

The poor performance of South African school students in international comparative studies such as the PISA and TIMMS studies is well known. This poor performance unfortunately not only occurs when it comes to international studies. Year after year the outcome of the Grade 12 mathematics National Senior Certificate (NSC) causes the alarm bells to ring and results have remained consistently low over the last three years (Department: Basic Education, 2011, p. 58). Universities, NGOs, government and the private sector are all concerned and have made many costly interventions to try and improve the results, but with almost no effect. The question is why? The first immediate reaction is to blame the curriculum. However, in a comparative study by Umalusi and Higher Education South Africa (HESA) (2010), it was found that the National Curriculum Statement (NCS) and accompanying national examinations compare well with other international curricula.

That leads us to the question: How do teachers implement this curriculum in their classrooms? No matter what the quality of the curriculum documents, the teachers who should implement them filter and change them, and ultimately determine what is implemented. Van den Akker (2003, p. 3) differentiates between the intended, the implemented and the attained curriculum:

- *Intended: as formally specified in curriculum documents and curriculum material and envisaged by a particular paradigm or philosophy.*
- *Implemented: as perceived and interpreted by teachers and put into operation through teaching and learning.*

- *Attained: as experienced and perceived by learners and through the quality of learning as indicated by learner outcomes.*

There is always a gap between the envisioned and the implemented curriculum. Regardless of improvements in the educational system, what happens in the classroom ultimately determines the size of the gap between the intended and implemented curriculum (Umalusi, 2008). Lolwana (2007) explains that:

> *Good education happens in the classroom, yet this is the 'black box' in the education system. What occurs in this space is the most difficult to quantify and change. Classrooms are secret places and learners often pay the price of this privacy. South African learners perform poorly compared to many countries with lesser economic advantage.*

Kilpatrick (2001) found that proficiency in mathematics depends on the extent to which teachers are able to consistently help students to learn worthwhile mathematical content. The nature, depth and breadth of the activities that the learners are engaged in determine their understanding of and performance in mathematics.

This study focuses on the gap between the intended curriculum (as specified in curriculum documents) and the implemented curriculum (as put into operation by the teachers), measured in terms of classroom activities. The research question is: To what extent do Gauteng FET teachers implement the South Africa Grade 10 to 12 (FET) curriculum?

THE INTENDED CURRICULUM

South Africa has one national curriculum as prescribed by the Department of Education. The National Curriculum Statement (NCS) and accompanying Subject Assessment Guidelines (DoE) compare well with other international curricula (Umalusi & HESA, 2010). There are four learning outcomes:

- Learning Outcome 1: Number and Number Relationships
- Learning Outcome 2: Functions and Algebra
- Learning Outcome 3: Space, Shape and Measurement
- Learning Outcome 4: Data Handling and Probability

The following documents elaborate on the intended FET Mathematics curriculum:

- National Curriculum Statement: Grades 10-12 (General) Mathematics
- Subject Assessment Guidelines – Mathematics
- Learning Programme Guidelines – Mathematics
- Mathematics Examination Guidelines: Grade 12 (2009)
- Gauteng Mathematics Work Schedule for Grade 10-12 (2009)
- Exemplar Examinations for Grade 10 to 12

These documents explain what is required from the FET (Grade 10 to 12) curriculum in terms of content coverage, content depth, the progress between and within grades, and how and what will be assessed in the final Grade 12 examination. The Gauteng Department of Education provides a detailed Work Schedule to each FET mathematics teacher. These guidelines explain to the teachers what to do and when to do it in order to cover all the topics. The Subject Assessment Guidelines for Mathematics and the Mathematics Examination Guidelines for Grade 12 explain the level of complexity of the mathematical questions in examinations. The guidelines use Bloom's taxonomy to classify assessment tasks into different levels of cognitive demand, but also use some of the taxonomical descriptors of the 1999 TIMSS Mathematics survey. The level of complexity of the mathematical questions is divided into four categories, namely knowledge (25%), routine procedures (30%), complex procedures (30%) and problem solving (15%). The knowledge category includes the use of algorithms, recall, using simple mathematical facts and formulae. The Subject Assessment Guidelines for Mathematics (DoE, 2008) explain and describe routine procedures as simple applications and calculations that require many steps. A complex procedure involves problems that are mainly unfamiliar and do not have a direct route to the solution. The fourth category about solving problems is mainly about "solving non-routine, unseen problems by demonstrating higher level understanding and cognitive processes" (DoE, 2008, p. 13). From this we can assume that teachers should have a clear picture of the intended curriculum.

CONCEPTUAL FRAMEWORK

This study investigates the gap between the intended and the implemented curriculum in terms of content coverage, the depth of coverage and curriculum pacing. This is not a new idea. In their study, Reeves and Muller (2005) use content coverage, content emphasis and curricular pacing to measure opportunity to learn. According to Winfield (1987), opportunity to learn may be measured by "time spent in reviewing, practicing, or applying a particular concept or by the amount and depth of content covered with particular groups of students" (p. 439).

Curriculum pacing

Gillies and Quijada (2008) explain that learning depends to some degree on time and effort, and warn that "without adequate time on task, no learning is possible" (p. 3). Abadzi (2007) found that:

> *… in low-income areas worldwide, from Brazil to Niger, only a fraction of the intended instructional time is used for learning tasks. Schools may close informally before or after holidays, start late in the day or end early. When teachers are present, they may be engaged in activities other than teaching.* (p. v)

This is also the case in South Africa. The Minister of Basic Education, Angie Motshekga (2009), warned that in some instances schools lose valuable teaching time because of the absence of teachers and because of incompetent principals. In a local pilot study done by Carnoy et al. (2008) it was found that teacher absenteeism is a significant problem in more than 70% of the schools and that teachers devote on average 3.2 hours to actual teaching during a school day.

The problem is not the availability of time, but rather the effective use of time for teaching and learning. A teacher is responsible for classroom organisation and the effective use of time inside the classroom. Some researchers use the term 'time-on-task', which refers to the amount of time that learners are actively engaged in learning (Gillies & Quijada, 1998). Abadzi (2007) warns that:

> *… even modest time wastage may result in significant student losses and that this can result in teachers just lecture in a hurry rather than analyze the content and use the teaching aids provided to schools, or they may omit parts of the curricula.* (p. 33)

The wasteful and inefficient use of time will result in less teaching time and will make it impossible for teachers and students to cover the curriculum.

Curriculum coverage

Taylor (2008b) identifies curriculum coverage as the biggest problem in South Africa and is of the opinion that this, together with teachers' poor content knowledge, must be addressed to improve the results. A study by the World Bank (Abadzi, 2007) reveals that achievement of learning outcomes should not be expected without sufficient teaching and practice opportunities. In his reanalysis of the results of the First International Study of Mathematics achievement, Fletcher (1971) concludes that achievement "is virtually synonymous with "coverage" across countries" (p. 145). The question is not only how much time is spent on learning activities and whether the curriculum was covered, but of major importance is the quality of such coverage. To develop a deep understanding of mathematics, enough work must be done on an appropriate level.

Content depth

The levels of content complexity of the activities in which learners engage are closely related to their understanding of mathematics (Carnoy et al., 2008; HSRC, 2008). Kilpatrick et al. (2001) identified five strands of mathematical proficiency, namely procedural fluency, conceptual understanding, adaptive reasoning, strategic competence and productive disposition. These strands represent a range of levels of cognitive demand. The researchers elaborated on the meaning of adaptive reasoning, namely the capacity for logical thought, reflection, explanation and justification. They also explained that strategic competence is about the ability to formulate, represent and solve mathematical problems. According to Webb (2010), there is general agreement that learners need to engage and experience, and that they should do mathematics and activities from a range of levels of cognitive demand. In fact, the TIMSS classroom video study illustrated that the learners of higher performing countries work more frequently on activities that require higher levels of cognitive demand (Stigler & Hiebert, 2004). Webb (2010) concludes that content complexity has been shown to relate to student performance.

RESEARCH METHOD

The current study comprises both qualitative and quantitative research. Time on task, curriculum coverage and level of cognitive demand of teaching and learning are aspects that are difficult to measure. The presence of researchers in the classroom may change teacher and learner behaviour and may sacrifice the reliability and validity of measurements in the process. Learners' workbooks proved to be a more reliable source of information for answering the research question. In each of the participating schools in Gauteng, the best two workbooks of a Grade 10 to 12 learners was selected by his/her own teacher and then submitted to be copied and analysed. This exercise was carried out towards the end of the last quarter, before the final examinations.

Participants

The study concerned was done in Gauteng, which was also the province with the highest Grade 12 pass rates in 2010, namely 78.6%. Schools are divided into quintiles according to their socio economic status, with the poorest schools in Quintile 1 and the wealthier schools in Quintile 5. Schools in Quintiles 1 and 2 are typically situated in informal settlements and townships; schools in Quintiles 3 and 4 are typically situated in suburbs and townships, and schools in Quintile 5 are typically situated in cities and suburbs. A sample of 18 schools in Gauteng, stratified by district and quintile, was randomly selected. For referencing purposes, the sample schools will be referred to as schools A to R. We only managed to make 13, 15 and 15 copies of the Grade 10, 11 and 12 workbooks respectively in the 18 schools (see Table 1).

Data collection and analysis

In order to answer the research questions, the two best workbooks (as selected by their own teacher) in grades 10, 11 and 12 of each of the schools in the sample were copied and analysed towards the end of the last quarter, before the final examinations. The assumption was that the best workbooks, as selected by the teacher, would contain all the class work and homework activities that the teacher had given to their learners. The best two workbooks were supposed to reflect the learning opportunities as created by the teacher. Eight expert mathematics teachers were selected to capture the data from the workbooks.

Instruments

Three similar instruments were developed to capture the data from the workbooks, one for each FET level. Subject Assessment Guidelines for Mathematics and the Mathematics Examination Guidelines for Grade 12 in 2009 were used to identify the topics and subtopics in each of the Grade 10 to 12 levels. A data collection instrument loosely based on the ideas of Reeves and Muller (2005) was designed. The following information from each topic or subtopic was recorded:

- The number of days that the learners spent working on a topic compared to the number of days as suggested by the Department of Education.
- The extent of curriculum coverage measured by the number of exercises done per prescribed topic and subtopic as suggested by the NCS.
- The number of exercises done on different levels of cognitive demand (knowledge, routine procedures, complex procedures, problem solving) for each topic and subtopic.

Together these elements told us more about how and to what extent teachers were implementing the NCS in their classrooms.

RESULTS FROM DOCUMENT ANALYSES

Given all the documentation and information provided by the DoE, the teachers should know and understand the intention of the curriculum in terms of content coverage, content depth, coherence and time needed to teach the different topics. From a discussion with the eight expert teachers, however, it seemed that there was some difference of opinion about the depth in which some topics should be covered. A few examples are logarithms in Grade 12, solving quadratic and exponential equations in Grade 11 (k-method, roots and complexity of fractions), and the inclusion of Euclidian geometry (triangles and quadrilaterals) in the compulsory part of the Grade 10 curriculum.

RESULTS FROM WORKBOOKS

We were able to identify an average of 48.5, 56.6 and 53.9 active ‘workbook-days’ in the entire school year from the sample schools for grades 10, 11 and 12 respectively. Table 1 illustrates the number of days that the students actually worked in their books in each of the schools. The lowest number of workbook-days (days students worked in their workbooks) that was calculated was 16 and the highest number was 128 days out of the school year.

Table 1: The number of ‘workbook-days’ per grade

School	Quintile	Workbook-days Grade 10	Workbook-days Grade 11	Workbook-days Grade 12
A	1	75	17	49
B	1	46	25	35
C	1	26	13	44
D	1	18	48	16
E	2	34	57	47
F	2	63	96	61
G	2	–	–	70
H	2	–	–	–
I	3	53	47	–

J	3	43	39	43
K	3	23	21	–
L	3	–	–	–
M	4	25	72	44
N	4	44	35	60
O	4	26	76	–
P	5	100	128	90
Q	5	101	114	105
R	5	50	61	37
Mean		48.5 days	56.6 days	53.9 days

Grade 10 workbook results

From the data above it is clear that we could obtain Grade 10 data for 15 schools only and an average of 48.5 active Grade 10 workbook-days was identified in these 15 schools. Figure 1 below shows the average number of days that these schools spent on each topic compared to the approximate number suggested by the Gauteng Department of Education. The average number of days spent on a topic was in each case less than the suggested number, except for exponents and number systems. Factorisation and products (simplification) enjoyed acceptable coverage in terms of the number of days spent on these topics.

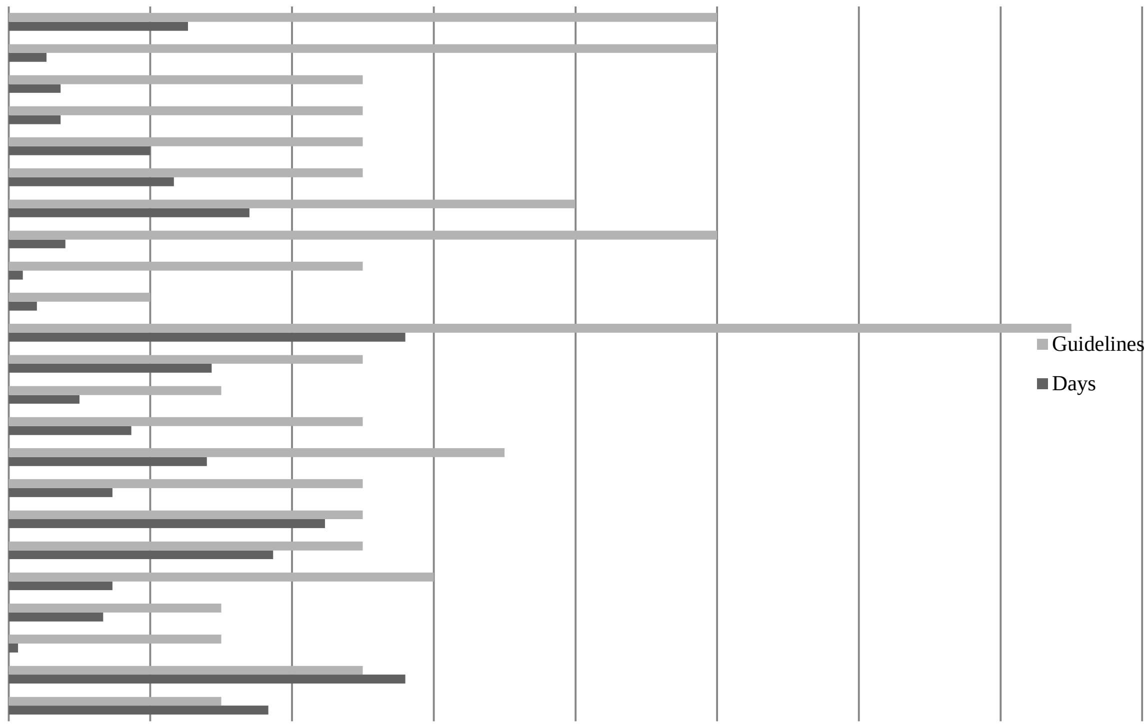

Figure 1: Average number of days that Grade 10s spent on each topic versus GDE guidelines

Some of the most problematic sections were geometry, trigonometry and modelling. In general, the traditional Paper 2 topics were neglected. This could be expected to result in a knowledge gap and make it difficult for these learners to understand the Paper 2 content in Grade 11 and 12. Table 2 and Figure 1 tell the same story, which seems quite logical – the fewer days spent on a topic, the fewer exercises could be done on the topic. Table 2 gives us an indication of the width and the depth of curriculum coverage and illustrates the limited number of exercises done on a higher level of cognitive demand in all topics.

Table 2: Average number of exercises done in Grade 10 topics on the different levels of cognitive demand

Grade 10 topics	**Knowledge**	**Routine procedures**	**Complex procedures**	**Problem solving**
Number systems	7	7	1	0
Exponents	15	12	3	1
Scientific notation	0	0	0	0
Surds (roots)	3	6	0	0
Number patterns	3	1	0	0
Products	7	7	4	0
Factorisation	8	19	3	1
Fractions	3	4	1	0
Equations	3	7	1	0
Simultaneous equations	1	3	0	0
Inequalities (linear)	1	2	0	0
Financial mathematics (growth)	3	3	0	0
Graphs: Linear, parabola, hyperbola, exp	7	8	1	0
Exchange rates	0	1	0	0
Modelling	0	0	0	0
Volume & surface area	1	1	0	0
Analytical geometry	6	7	0	0
Transformations	2	2	0	0
Trigonometry: Introduction	9	6	0	0
2D	1	1	0	0
Geometry: Triangles	2	1	0	0
Quadrilaterals	0	1	0	0
Data handling	4	3	1	0

According to the Grade 10 assessment guidelines, learners must be able to simplify expressions and solve equations with fractions with only one term in the denominator. However, teachers in this study were still using two and three terms, as prescribed by the previous curriculum.

Grade 11 workbook results

An average of 56.6 active Grade 11 workbook-days could be identified in 15 of the 18 sample schools. Figure 2 indicates the average number of days that Grade 11 learners in these schools spent on each topic, compared with the approximate number suggested by the Gauteng Department of Education. In each case the number of days found was less than the suggested number of days – except for exponents, fractions and quadratic equations. The reason for the latter is that although the amount of content about these topics was scaled down from the previous curriculum to the NCS, a number of teachers were still teaching the previous curriculum content.

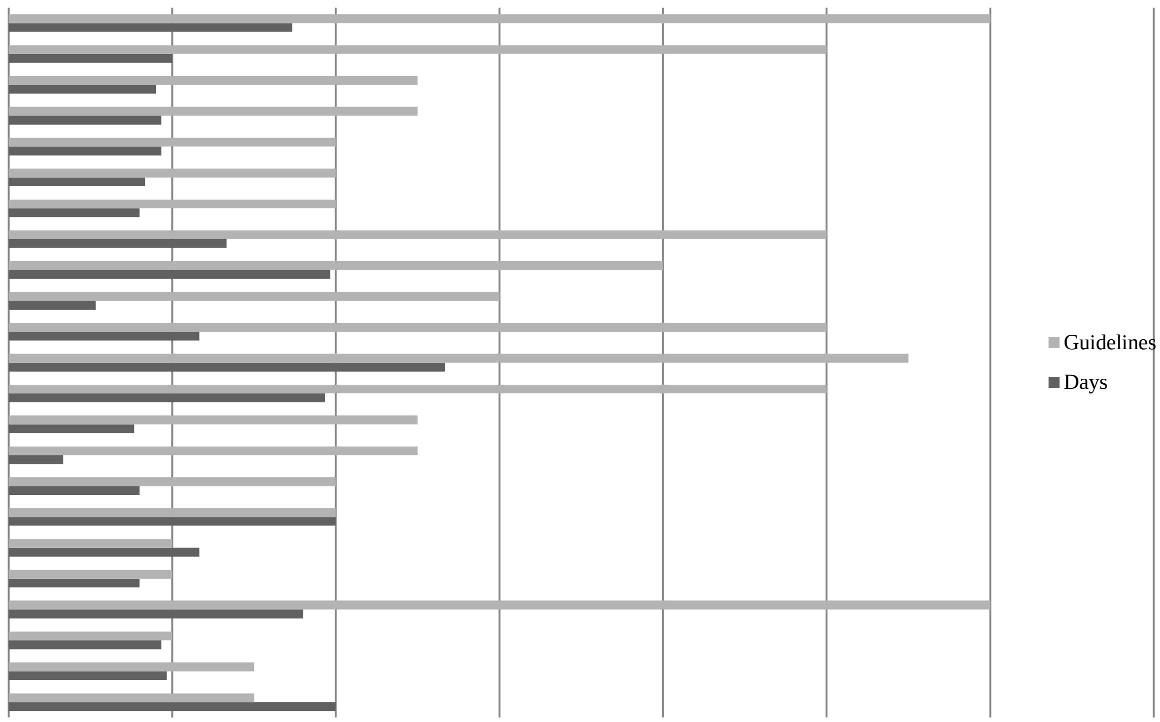

Figure 2: Average number of days that Grade 11s spent on each topic versus GDE guidelines

Figure 2 shows that the most problematic topics were data handling; the application of sine, cosine and area rules in two dimensions; transformations; volume and surface area; linear programming; inequalities; modelling, and number patterns. Table 3 indicates how widely and deeply the curriculum was covered and illustrates the limited number of exercises done on a higher level of cognitive demand in all topics.

Table 3: Average number of exercises done in Grade 11 topics on the different levels of cognitive demand

Grade 11 topics	Knowledge	Routine procedures	Complex procedures	Problem solving
Exponents	4	12	3	1
Surds	2	5	1	0
Number system	3	4	0	0
Number patterns	2	4	1	0
Completion of square	1	1	1	1
Fractions	1	6	3	1
Quadratic equations	3	8	2	0
Simultaneous equations	1	2	1	0
Modelling	0	0	1	0
Inequalities	1	3	1	0
Financial mathematics	3	7	2	1
Graphs: Linear, parabola, hyperbola, exp	3	9	2	1
Linear programming	2	2	1	1
Volume & surface area	1	2	1	0
Analytical geometry	6	5	1	0
Transformations	2	5	0	0
Trigonometry: Special angles	3	5	1	0
Identities	1	3	1	0
Reduction formula	4	4	1	0
Graphs	1	2	1	0
Equations	1	5	0	0
Sin, cosine & area rules	1	3	0	1
Data handling	4	6	1	0

Teachers were found to be wasting up to twelve days on topics and subtopics that were in the previous curriculum. They spent time on complex and technical calculations that were not stipulated in the new curriculum and that were not assessed according to the Grade 12 Examination Guidelines. When solving quadratic equations, teachers were still focusing on the use of the k-method, doing complex square roots, and solving fractions with more than two terms in the denominator. Modelling problems where learners had to use the methods of quadratic equations in solving real-life problems were almost non existent. Only two of the fifteen schools did some of these problems. Teachers did not teach the Grade 11 topics that were not directly assessed in the final Grade 12 exams, e.g. modelling, volume and surface area, and completion of the square. According to the assessment guidelines, volume and surface area are included in the Grade 11 curriculum and are assessed in Paper 2, but in Grade 12 it is only assessed in Paper 1 and as part of the application of differentiation. That may be the reason why seven of the fifteen schools did not bother to do the topic.

Grade 12 workbook results

Figure 3 compares the average number of days that learners in the sample schools spent on each maths topic with the approximate number suggested by the Gauteng Department of Education. The average number of days spent on a topic equals in general half the suggested number of days, except for one topic, namely logarithms. A possible reason for this is that the amount of content about logarithms had been scaled down from the previous curriculum to the NCS (introduced in 2008 for Grade 12) and a number of teachers were probably still teaching the old curriculum content.

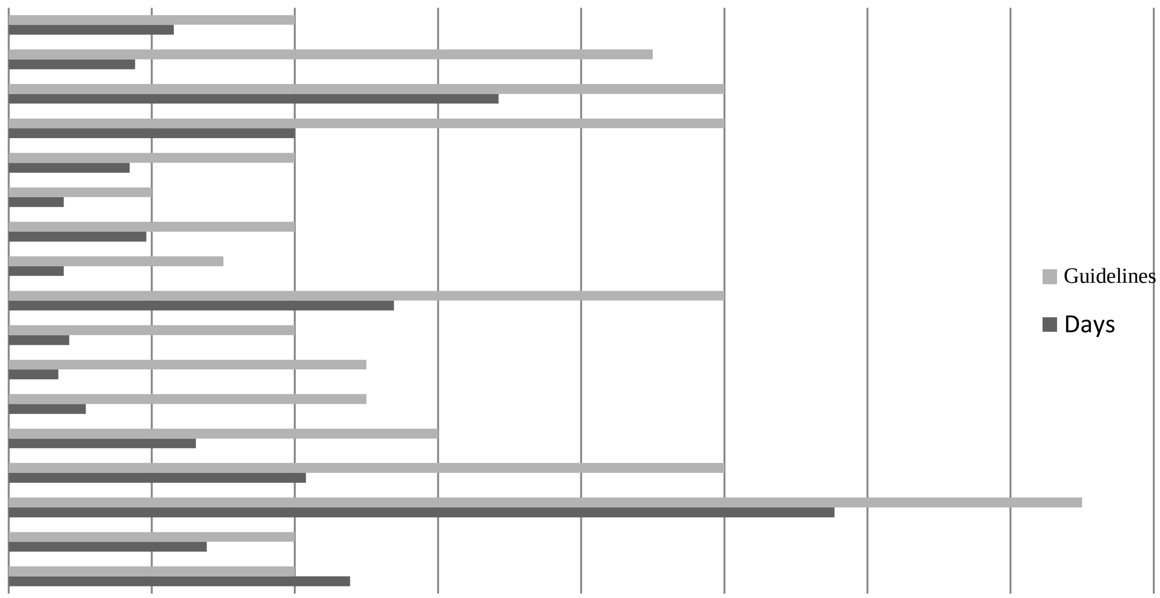

Figure 3: Average number of days that Grade 12s spent on each topic versus GDE guidelines

The study revealed that the most problematic sections were the application of sine, cosine and area rules in three dimensions; linear programming; applications of differentiation; inverse of functions; transformation of graphs, and properties of graphs. It is quite significant that of these problematic topics the only new NCS topic is transformation of graphs.

The logical conclusion referred to above was again confirmed in this case: the fewer days spent on a topic, the fewer exercises could be done on the topic. Table 4 shows the width and the depth of curriculum coverage and points out how few exercises were done on a higher level of cognitive demand in all topics.

Table 4: Average number of exercises done on the different levels of cognitive demand

Grade 12 topics	Knowledge	Routine procedures	Complex procedures	Problem solving
Logarithms	7	17	0	0
Cubic equations & factors	2	10	0	0
Patterns & sequences	4	34	3	1
Finance: Annuities	1	9	2	0
Functions & graphs	2	3	0	0
Properties of graphs	1	2	0	0
Transformation of graphs	1	2	1	0
Inverses	2	2	1	0
Differentiation (theory)	5	12	1	0
Max & min problems	0	1	0	1
Cubic graphs	0	3	1	0
Tangents	1	1	0	0
Linear programming	0	1	1	1
Analytical geometry	4	8	1	0
Compound & double angles	5	15	3	1
Solving triangles in 3D	1	2	1	1
Transformations: Rotations	2	6	1	0

Table 4 demonstrates the absence of higher-order questions and shows that the majority of teachers and learners rarely engaged in problem-solving activities. The main focus in the classrooms was on procedural fluency and there was almost no evidence of adaptive reasoning and strategic competence, except for the best performing schools (M, P and Q).

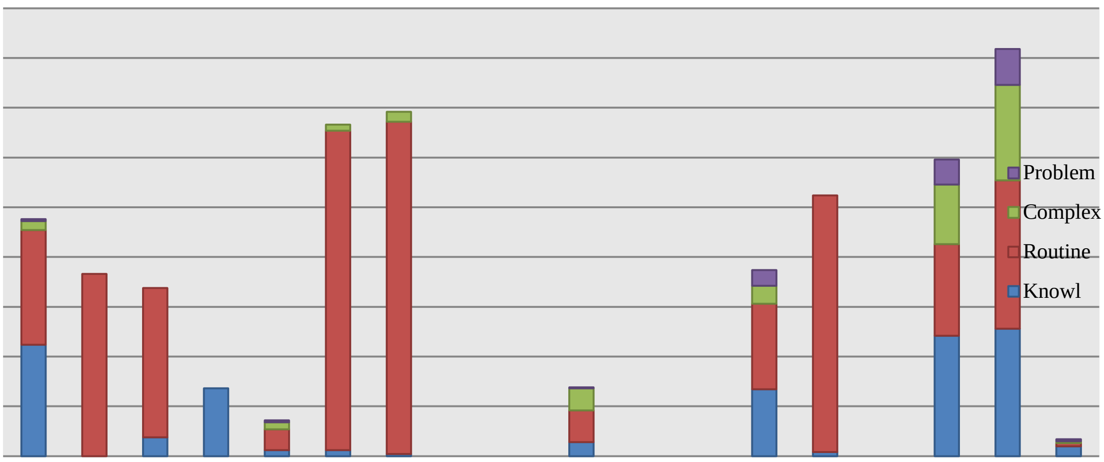

Figure 4: Total number of exercises done during the school year on different levels of cognitive demand

General comments made by the expert teachers about the Grade 12 learners' workbooks:

"Teachers are wasting time on topics and sub topics that were in the old curriculum. Up to fifteen days was spending [sic] *on old curriculum work in some of the schools. A number of the teachers are still doing logarithms in detail, including all the laws and solving logarithmic equations."*

"They are also wasting time on doing technical questions about the remainder and factor theorem. The teachers in general do not understand the spirit of the new curriculum with the emphases [sic] *on understanding and application (and not long technical calculations)."*

After analysing the workbooks of a school, one of the experts wrote: "*They are doing the work that is not examined in great detail but the important work is neglected.*"

DISCUSSION

As indicated earlier, we were able to identify an average of 48.5, 56.6 and 53.9 active 'workbook-days' in the entire school year from the sample schools for grades 10, 11 and 12 respectively (Table 1). The number of official school days per annum is 199, which includes the examinations. If we subtract 49 days for assessment activities, there are still 150 days left for learners to work in their workbooks. Different factors may have contributed to the extremely low number of workbook-days and curriculum coverage found in the study, for example teachers' absence, teachers not in classes, time spent on assessment tasks, and ineffective teaching methods. Ngoepe and Treagust (2003) found that teachers in the South African schools that they visited wasted much time writing down solutions to problems on the chalkboard for learners to copy. Also, instead of referring to textbooks, teachers wrote all the class work on the chalkboard and gave homework exercises orally. This way of teaching not only proved to be ineffective in terms of conceptual development, but also wasted valuable learning time.

The small number of active 'workbook-days' resulted in very limited curriculum coverage in almost all of the topics. Teachers spent more time on the topics that they knew and felt confident to teach, e.g. exponents and logarithms. They tended to avoid the more complex topics such as modelling, geometry and applications of trigonometry. The expert teachers generally commented that teachers were wasting time on subtopics that had been in the previous curriculum and on technical questions. In general, they did not understand the spirit of the curriculum where the emphasis was placed on understanding and application. The most neglected topics on all levels of cognitive demand in the Grade 12 curriculum were functions (graphs, properties of graphs, transformation of graphs); applications of differentiation (maximum and minimum problems, graphs of cubic functions, tangents); linear programming, and solving triangles in three dimensions. From this list of topics we can see that except for functions, all the other topics involve the application of mathematics in real life (modelling). In the purpose statement of the FET NSC (2003: 10), the document emphasises the importance of mathematical modelling:

> *An important purpose of Mathematics in the Further Education and Training band is the establishment of proper connections between Mathematics as a discipline and the application of Mathematics in real-world contexts. Mathematical modelling provides learners with the means to analyse and describe their world mathematically, and so allows learners to deepen their understanding of Mathematics while adding to their mathematical tools for solving real-world problems.*

To develop a deep understanding of mathematics, enough work must be done on all levels of cognitive demand. The results, however, show that the main focus of the workbook activities was on the development of knowledge and routine procedures. A minimum of work was done on the higher levels of cognitive demand. In fact, it was only in the three top performing schools in the sample that any evidence was found of work done that required problem solving. Higher-order thinking skills are essential basic skills for the twenty-first century. Confrey and Lachance (2000) explain why:

> *These skills [computational skills] allow students to secure jobs and to become informed citizens in an industrial society. However, with advances in technology, such computational skills are no longer as important. Instead, students need to develop critical-thinking skills to interpret data appropriately and to use technology to solve more complex problems. Thus, changes in our society have led to a change in what we value in mathematical skills.* (p. 232)

The NCS Mathematics curriculum for Grades 10 to 12 focuses on the development of higher-order thinking skills. One of the critical outcomes of this curriculum is that school students should be able to identify and solve problems and make decisions using critical and creative thinking (National Department of Education, 2002). It is not only in South Africa that the development of higher-order thinking skills is receiving considerable attention. According to Yelland (1997, p. 1), the quest for higher-order thinking in curricula is an international trend: "…educational programmes that support the development of higher-order thinking skills are in evidence around the globe". Higher-order thinking skills are also important for independent life-long learning:

> *The importance of having students develop good critical and creative thinking abilities has to do with the foundations needed for a democracy and with the tools needed for independent and life-long learning (Saskatchewan Education, 1988).*

The inadequate amount of time spent on, and the limited number of exercises done in functions, are also a point of concern because functions are a unifying concept across the entire mathematical curriculum. In higher education, functions are used in almost every branch of mathematics.

CONCLUSION

Looking at the results of our study, the poor performance of South African learners in both international comparative studies and in the National Senior Certificate (NSC) is understandable. It is not only the absence of activities on higher levels of cognitive demand that is a concern, but the fact that the majority of daily work done in the maths classroom does not include enough exercises to master even the most basic level of mathematical procedural proficiency. The South African Minister of Basic Education, Angie Motshekga (2009) admitted in Parliament that "[t]he culture of teaching and learning has, for all intents and purposes, disappeared in most rural and township schools".

First World countries are concerned about the gap between the implemented and attained curriculum. Their concern is about the effectiveness of teaching and learning and about the creation of a more learner-centred environment. The challenges in South Africa, however, are curriculum coverage and basic work ethics (***being in class, on time, teaching for seven hours a day***). Working less than 60 days per year cannot lead to good results. The main focus of interventions in South Africa, at this stage, should be on narrowing the gap between the intended and implemented curriculum, rather than creating a more learner-centred environment.

The question is – how can we address the problem and where do we start? The HSRC (2008) explains that: "[e]ven though the level of cognitive demand is an aspect more related to the learner, it is the teacher that controls and directs the required level for his or her students". Future interventions to improve the achievement of South African learners should focus on their teachers. The President of South Africa, Jacob Zuma (2010, as quoted at a meeting of the National Council of Provinces), took a step in this direction by emphasising: "Teachers must be in class, on time, teaching for seven hours a day".

RECOMMENDATIONS

- Provide in-class support for teachers (using expert teachers).
- Hold teachers accountable for high quality curriculum coverage.
- Let the learners of poor performing schools use structured workbooks (with prescribed days and dates) to help their teachers to cover the entire curriculum in depth.
- Provide the learners with a toll-free number to report poor curriculum coverage.

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Acknowledgements

This study formed part of a larger study carried out at the Centre for Evaluation and Assessment (CEA) in the Faculty of Education of the University of Pretoria. I would like to thank all the teachers, fieldworkers and colleagues for their participation and contributions.

THE VAN HIELE LEVELS AS CORRELATE OF STUDENTS' ABILITY TO FORMULATE CONJECTURES IN SCHOOL GEOMETRY

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This collective case study employed the van Hiele model of geometric proficiency to explore grade 12 learners' abilities to formulate conjectures in circle geometry in selected Nigerian and South African schools. The study further interrogated the relationship between the van Hiele levels and a learner's ability to formulate geometry conjectures. The sample comprised a total of 48 learners, 24 each chosen from a high school in Lagos, Nigeria (NS) and 24 from a 'township' high school in the Eastern Cape, South Africa (SAS). The participants were selected using stratified sampling techniques. The instruments for data collection were the Conjecturing in Plane Geometry Test (CPGT) and the Van Hiele Geometry Test (VHGT). Although the participants generally obtained a low mean score (36.5%) in the CPGT, indicating that they had difficulties with formulating conjecture in circle geometry, learners from the SAS subsample were, nonetheless, more successful than their counterparts from the NS, since the difference between the mean score of the NS learners (20.68%) and that of the SAS learners was statistically significant ($p<0.05$). For both subsamples, learners' scores in the CPGT positively correlated with their VHGT scores (for NS, $r = 0.32$ and for SAS, $r = 0.55$), indicating that the van Hiele levels are positively related to a learner's ability to formulate conjectures in circle geometry. Based on these results, some recommendations are offered.

INTRODUCTION

No curriculum prescription of the body of knowledge that students are expected to learn in mathematics can be said to be complete without specifying how best the students might be apprenticed into acquiring that knowledge. This is because learning, especially school-based learning, is necessarily linked to teaching. This should explain why current reforms in mathematics curricula across the world tend to have focused more on "teaching mathematics better" than on "teaching better mathematics" that characterised curriculum reforms during the new maths era of the mid 1950s to mid 1970s (Kilpatrick, Swafford & Findell, 2001, p. xiv). Although geometry, in one form or another, continues to be one of the most cherished components of school mathematics the world over (Hart & Picciotto, 2001; Atebe & Schafer, 2011), it is, perhaps, also the most *troubled* in terms of curriculum reformations regarding its contents and its didactics. The reintroduction of Euclidean geometry currently being contemplated in the CAPS (Curriculum and Assessment Policy Statement) document in South Africa, attests to this hypothesis.

Shannon (2002, p. 26) asserts that the geometry that was "based [strictly] on the Euclidean system whereby knowledge of shapes was derived almost exclusively from a set of axioms, using deductive reasoning" alone, was beneficial only to "the top 20% of pupils in secondary school". Because of the negative results that were recorded over many years of it being taught in secondary education (van Hiele, 1999), Euclidean geometry has been criticised as too formal, too complicated and even too difficult. As a consequent, many countries (e.g. Russia, the US, and the Netherlands) advocated reforms in approaches to school geometry (Allendoerfer, as cited in Atebe, 2008). The reforms that took place in many countries reflected for the most part changes in didactics in the light of the research conducted in the late 1950s by two Dutch mathematics educators, Pierre van Hiele and his wife, Dina van Hiele-Geldof. In Russia, for example, results from the van Hieles' research have influenced major changes in the country's mathematics curriculum with recorded improvements in students' understanding of school geometry (Fuys, Geddes & Tischler, 1988).

In Nigeria and South Africa, no major reform of the geometry curricula (especially at the senior secondary level) appears to have been done along the outcomes of the van Hiele research[5] (De Villiers, 1997; 2010). Nevertheless, a cursory examination of these countries' mathematics curricula reveals that while each still retains Euclidean geometry as a core component of the curriculum, they tend to have offered alternative approaches that depart from the rather slavish adherence to Euclid's postulational approach that perplexed its teaching and learning for over two millennia (Bell, as cited in Atebe, 2008). In South Africa, for example, the National Curriculum Statement (NCS) Grades 10-12 (General) for mathematics tends to have favoured a didactical approach that has as its main focus investigations leading to enhanced students' abilities in stating and proving conjectures in geometry. This is lucidly exemplified in the statement that the study of space, shape and measurement (Learning Outcome 3) should enable the learners, among others to:

- explore relationships, *make and test conjectures* (emphasis added) and solve problems involving geometric figures and geometric solids
- *investigate* (emphasis added) geometric properties of 2-dimensional ... figures in order to establish, justify and prove conjectures

South Africa, Department of Education (DoE) (2003, p. 13)

But are the grade 12 learners in Nigeria and South Africa acquiring the all-important skills of formulating conjectures in geometry as mandated by the curricula? Do students' van Hiele geometric thinking levels relate to their ability to formulate conjectures in school geometry? As its main goals, this study intends to provide answers to these questions, by drawing on the van Hiele (1986) model of students' thought levels in school geometry. It is envisaged that mathematics educators will

[5] In South Africa, Feza and Webb's (2005) analysis of the geometry curriculum for the General Education and Training (GET) band tends to indicate that it reflects, in general, the van Hiele levels

find the results of this study significant in terms of assisting them in making decisions about their instructional design and delivery.

THE THEORY AND RELATED RESEARCH

The problem of children's difficulties with school geometry began to receive formal empirical and curricular attentions in many countries (e.g. the Netherlands, Russia and the US) as a result of the research conducted by a husband-and-wife team of Dutch mathematics educators, Pierre van Hiele and Dina van Hiele-Geldof, As high school teachers, the van Hieles noticed with disappointment the difficulties that their learners were having in geometry. Consequently, they completed companion doctoral dissertations (that culminated into what is today known as the van Hiele theory) at the University of Utrecht in 1957, with their respective foci on how schoolchildren learn mathematics and how to help them through instruction (Fuys et al., 1988).

The van Hiele theory identifies a sequence of five hierarchical thinking levels in learners' acquisition of geometric concepts starting with the recognition of shapes and culminating in being able to write a formal proof in geometry (Clements, 2004). The theory further offers a well sequenced five 'phases of learning' to assist teachers help their learners to make progress through the levels (Atebe, 2008). However, in this paper, only the van Hiele levels will be explicated as they provide a more useful framework for analysing and interpreting the results of this study.

The van Hiele levels

Consistent with current trend in the literature (e.g. van Hiele, 1986; 1999; Clements, 2004; De Villiers, 2010), this study made use of the 1–5 numbering scheme for the van Hiele levels. Therefore, all references made to research that used the van Hiele's original 0–4 numbering scheme have been adapted to the 1–5 numbering scheme. The five van Hiele geometric thinking levels, in order, are as follows:

Visual level: Learners recognize geometric shapes as a whole. They are able to visually recognize and name triangles, rectangles and parallelograms etc, but they do not explicitly identify the properties of these shapes.

Descriptive level: Learners begin to reason about a geometric shape in terms of its properties. They are able to recognize and describe properties of shapes using the correct terminology, but they do not yet understand the interrelationships between these properties and between different shapes. Class inclusion is not yet understood, e.g. they will reject the inclusion of a rectangle in the class of parallelograms.

Ordering level: Learners logically order the properties of shapes by short chains of deductions and now understand the interrelationships between shapes (e.g. class inclusions). They are able to formulate economically correct definitions for shapes, but the roles of axioms, theorems and proofs are not yet understood.

Deductive level: Learners begin to develop longer sequence of statements deducing one from the other to justify observations. They now understand the significance of deduction, the role of axioms, theorems and proofs. Necessary and sufficient conditions are understood, but the necessity for rigour is not yet recognized.

Rigorous level: Learners now reason formally about mathematical systems, understand the necessity for rigour and are able to make abstract deductions. Non-Euclidean geometries can be studied.

THE RESEARCH QUESTIONS

This study was guided by two main questions:

- To what extent are the selected grade 12 learners able to formulate conjectures in circle geometry?
- Is there a correlation between learners' van Hiele levels and their ability to formulate conjectures in circle geometry?

METHOD

The research design is a collective case study (Stake, 2000) conducted in Nigeria and South Africa, and it employs quantitative methods. According to Jackson (1995), a quantitative research technique, typically attempts to describe relationships among variables statistically and to present a numerical analysis of the social relationships being studied.

The sample

The sample for this study comprised a total of 44 grade 12 Nigerian and South African high school mathematics learners. Of the 48 learners, 24 were drawn from a public school in Lagos, Nigeria and the other 24 from a comparative (e.g. school type, infrastructure, curriculum contents, etc) 'township' school in the Eastern Cape, South Africa. The sampling procedures involved were purposive – a focus on the grade 12 learners, stratified – subdividing learners into three ability levels (high, average and low achievers), and the fish-bowl technique – constituting the study sample with equal number of learners randomly selected from all three ability levels. The focus on the grade 12 learners came as the consideration that grade 12 represents a major transition (from secondary to tertiary education) in each educational system, and a point at which the 'fruits' of secondary education can be assessed. The findings from this study, thus, hold important implications for both countries in terms of the quality of students (cognitively speaking) who are accessing (or hoping to access) tertiary education in Nigeria and South Africa.

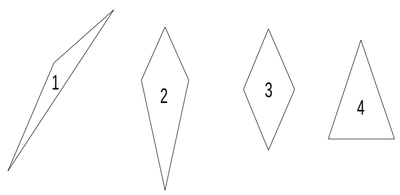
The instrument

Two major instruments were used for data collection in this study. These were the VHGT (van Hiele Geometry Test) and the CPGT (Conjecturing in Plane Geometry

Test). Following the development of the van Hiele theory of the levels of thought in geometry, experts and professional bodies have since developed achievement tests that can be used to measure the attainment of the levels among schoolchildren (Hoffer, 1983). One such test is the CDASSG (Cognitive Development and Achievement in Secondary School Geometry), which is widely used in the US (Usiskin, 1982). The VHGT used in this study is an adapted version of the CDASSG, which was originally designed to determine the van Hiele levels of geometric understanding of American schoolchildren. The CDASSG test items "were based on direct quotations from the van Hieles' writings and were piloted extensively" (Senk, 1989, p. 312).

The reason for adapting (rather than adopting) the CDASSG test was that learners do not think at the same van Hiele levels in all areas of geometry contents (Senk, 1989). Therefore, van Hiele (1986) and Senk (1989) suggest that studies that seek understanding of the thinking processes that characterize the van Hiele levels should be content specific. This suggests that as the CDASSG test was constructed, presumably, in accord with the US geometry curriculum, it made sense to adapt the test questions in ways that reflect the Nigerian and South African geometry curricular prescriptions. The VHGT was a multiple-choice test, and it comprised 4 subtests. Each subtest consisted of 5 items based on one van Hiele level. That is, there were in all 20 items in the VHGT, with items number 1-5, 6-10, 11-15 and 16-20 testing learners' attainments of van Hiele levels 1, 2, 3 and 4 respectively. Figure 1 represents sample items from subtests 1 and 4.

Question 1. Which of these are triangles?



A. All are triangles
B. 4 only
C. 1 and 2 only
D. 3 only
E. 1 and 4 only

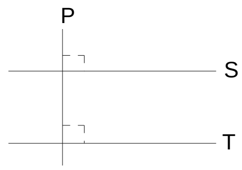
Question 17. Examine these statements.

i). Two lines perpendicular to the same line are parallel.

ii). A line perpendicular to one of two parallel lines is perpendicular to the other.

iii). If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines S and P are perpendicular and lines T and P are perpendicular.



Which of the above statements could be the reason that

Figure 1 Sample items from levels 1 and 4 subtests

The CPGT was in the form of a worksheet and was designed to explore learners' mathematical knowledge of circle geometry in three prescribed areas of their curriculum: Chord properties of a circle; Arc-angle properties of a circle; and Tangent properties of a circle. For the CPGT a constructivist investigative approach was adopted whereby learners were required to investigate (through geometrical construction) and discover properties and relationships (e.g. chord-angle relationship) in a circle. Such investigation and discovery, it was assumed, should enable the learners to draw simple inferences and formulate conjectures in circle geometry. The worksheet consisted of 10 investigations:

- Investigation 1 was to guide the learners to develop a conjecture that *the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord*.
- Investigation 2 was to guide the learners to develop a conjecture that *the line drawn from the centre of a circle perpendicular to a chord bisects the chord*.
- Investigation 3 was to guide the learners to develop a conjecture that *equal chords are equidistant from the centre of a circle*.
- Investigation 4 was to guide the learners to develop a conjecture that *equal chords subtend equal angles at the centre of a circle*.
- Investigation 5 was to guide the learners to develop a conjecture that *the angle which an arc of a circle subtends at the centre is twice that which it subtends at any point on the remaining circumference.*
- Investigation 6 was to guide the learners to develop a conjecture that *the angles in the same segment of a circle are equal*.
- Investigation 7 was to guide the learners to develop a conjecture that *the angle subtended by the diameter of a circle is a right angle*.
- Investigation 8 was to guide the learners to develop a conjecture that *the opposite angles of a cyclic quadrilateral are supplementary*.
- Investigation 9 was to guide the learners to develop a conjecture that *a tangent to a circle is perpendicular to the radius at the point of contact*.
- Investigation 10 was to guide the learners to develop a conjecture that *tangents to a circle from the same external point are equal in length.*

Although the CPGT is originally mine, important ideas from the interview schedules of Mayberry (1983) and Burger and Shaughnessy (1986), as well as the work of Siyepu (2005) were incorporated into its general format and method of questioning. For each investigation, participants were given guiding instructions that would enable them to complete the task[6]. For example, the step-wise instructions for investigations 3 and 4 are as shown in Figures 2 and 3 below.

[6] I have already indicated above and elsewhere (Atebe, 2008) that the Nigerian and South African geometry curricula have much in common. Hence, the tasks that constituted the CPGT were judged suitable for learners from both countries.

Investigation 3:

Step 1: Construct a circle, centre O and radius greater than or equal to 3cm.
Step 2: Construct two non-parallel congruent chords, AB and PQ which are not diameters, and are on different sides of the circle.
Step 3: Construct the perpendiculars from O to meet AB at M and to meet PQ at N.
Step 4: Measure OM and ON.
$|OM|$ = ……………………….; $|ON|$ = ………………………….
Compare your results with those of others near you and state your observation as a conjecture.
Conjecture: ……………………………………………………………………….....
………………………………………………………………………………………..
………………………………………………………………………………………..
………………………………………………………………………………………..

Space for your diagram.

Figure 2 Sample investigation task (investigation 3) for the CPGT

Investigation 4:

Step 1: Construct a circle, centre O and radius greater than or equal to 3cm.
Step 2: Construct two non-parallel congruent chords, AB and PQ which are not diameters, and on different sides of the circle.
Step 3: Draw radii, OA, OB, OP and OQ.
Step 4: Use your protractor to measure angles AOB and POQ.
$\angle AOB$ = ……………………, $\angle POQ$ = ………………………….
Compare your result with those of others near you and state your observation as a conjecture.
Conjecture: ……………………………………………………………………………
………………………………………………………………………………………...
………………………………………………………………………………………...

Space for your diagram.

Figure 3 Sample investigation task (investigation 4) for the CPGT

Test grading: Each correct response to the 20 items that constituted the VHGT was assigned 1 point. Hence, each learner's score ranged from 1–20 points. For the CPGT, I developed a 'marking scheme' with some general criteria for grading the responses of the learners based on the work of Senk (1985). In terms of these criteria, predetermined marks were assigned to specific elements in learners' responses that reflected the correct answer. The marking scheme was meant to reduce marker's subjectivity inherent in essay-type questions such as those of the CPGT. For both the VHGT and the CPGT, percentage mean scores were calculated for each of the learners.

RESULTS AND DISCUSSION

In order to provide an answer to the first research question, participants' performances in the CPGT were analysed. This analysis is presented in terms of percentage mean scores and in terms of an illustrative item analysis of participants' responses to the CPGT. For the purpose of this analysis and for ease of reference, NS and SAS were used to represent the participating schools from Nigeria and South Africa, respectively. The percentage mean scores in the CPGT were computed collectively for all the learners and separately for each of the NS and SAS subsamples.

The overall percentage mean score obtained by all the 42 learners (6 learners were absent on the day of testing) who wrote the CPGT was 36.5%. It should be recalled that the CPGT was designed primarily to explore learners' ability to formulate conjectures in circle geometry. Therefore, the low mean score obtained by this cohort of grade 12 learners in the CPGT could be interpreted as evidence that they had a poor knowledge of this aspect of school geometry. Since drawing inference and formulating conjectures are cognitive activities commonly associated with van Hiele levels 3 and 4 (Clements, 2004), the low mean score further indicates that the participants were not yet at these van Hiele levels of geometric proficiency. Results such as these should worry mathematics educators and other stakeholders, because formulating conjectures in geometry holds potentials to reinforce important skills such as observing patterns, identifying relationships, as well as being a first step towards formal geometric proofs – all of which contribute towards a learner's mathematical proficiency. Table 1 summarizes participants' performance in the CPGT for the NS and SAS subsamples.

Table 1 Percentage mean scores of participants in the CPGT

School	**N**	**Mean score**	**Std Dev.**	***t-value***	***df***	***p-value***

NS	22	20.68	22.05	- 3.72	40	0.0006
SAS	20	52.25	32.38			

As evident in Table 1, the mean score obtained in the CPGT by the grade 12 learners from NS was 20.68% and that of their counterparts from the SAS was 52.25%. The table further indicates that the difference between the means of the NS and SAS learners, in favour of the latter, is statistically significant (t=-3.72, 40df, p<0.05). This means that the grade 12 learners from the SAS performed significantly better than their peers from the NS.

The results shown in Table 1 need to be interpreted with caution. The very large values for the standard deviations suggest that constitutive of the sample were learners of varying cognitive abilities – some very high achievers and others very low achievers. Although having a sample with learners of mixed cognitive abilities was not unintended as explained earlier on, the percentage mean scores presented in Table 1 may have, nonetheless, been affected by this factor. The later explanation regarding these mean scores notwithstanding, the point that remains invariant is that, taken as whole, the participants in this study had difficulties with formulating conjectures in circle geometry.

AN ILLUSTRATIVE ITEM ANALYSIS OF LEARNERS' RESPONSES IN THE CPGT

Participants' responses were further interrogated with regard to the number of learners that were successful in performing specific activities in each of the 10 investigations that constituted the CPGT. Table 2 summarizes the results.

Table 2 Item analysis of learners' responses in the CPGT

Investigation No.	**Expected activity**	**No. successful**	
		NS (n = 22)	SAS (n = 20)
1	• To obtain, through own construction, angle 90° between the line drawn from the centre of a circle to the midpoint of a chord	14	15
	• To conjecture that the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord	7	14
2	• To obtain, through own construction, equal measure for the sides of a chord from the point of intersection of the perpendicular from the centre of a circle	13	14
	• To conjecture that the line drawn from the centre of a circle perpendicular to a chord bisects the chord	3	10
3	• To obtain, through own construction, equal measure for the lines drawn from the centre of a circle perpendicular to	8	15

	chords of equal length • To conjecture that equal chords are equidistant from the centre of a circle	1	4
4	• To obtain, through own construction, equal central angles for equal chords of a circle • To conjecture that equal chords subtend equal angles at the centre of a circle	8 0	14 6
5	• To obtain, through own construction, angle at centre = 2 x angle at circumference • To conjecture that the angle which an arc of a circle subtends at the centre is twice the angle which the same arc subtends at the circumference	8 2	10 10
6	• To obtain, through own construction, equal angle subtended by the same arc of a circle at two different points on the circumference • To conjecture that angles in the same segment of a circle are equal	7 2	13 11
7	• To obtain, through own construction, angle 90° for the angle subtended by the diameter of a circle • To conjecture that the angle in a semicircle is a right angle	8 1	12 8
8	• To obtain, through own construction, 180° as the sum of the opposite angles of a cyclic quadrilateral • To conjecture that opposite angles of a cyclic quadrilateral are supplementary	3 0	10 7
9	• To obtain, through own construction, angle 90° as the angle between a radius and a tangent at the point of contact • To conjecture that a tangent to a circle is perpendicular to the radius at the point of contact	5 1	15 7
10	• To obtain, through own construction, equal measure for two tangents to a circle drawn from the same external point • To conjecture that tangents to a circle from the same external point are equal in length	0 0	12 2

As stated earlier, investigation 1 of the CPGT (worksheet) was to guide the learners to form a conjecture that *the line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord*. Table 2 indicates that 14 (64%) of the NS learners were successful in obtaining, through their own construction, angle 90° between a chord and the line drawn from the centre of a circle to the midpoint of the chord. Only 7 (32%), however, were able to state their observation as a conjecture. Although Table 2 clearly indicates that formulating conjectures was generally difficult for the NS learners, it was particularly so for them with concepts that dealt with the tangent properties of a circle (investigations 9 and 10).

Although the inability of these learners to construct (draw) and take accurate measurements may have partly influenced their response pattern in these investigations, that alone cannot justify their poor performance in this learning area, since according to Siyepu (2005), they must (ought to) have had experiences in these skills (constructing, drawing and measuring) in their lower grades. Most of these learners could simply not perceive the interrelationships between the properties of a circle in the various investigations. The majority of them were not yet at van Hiele level 3, since according to the van Hiele theory, formulating (and testing/proving) conjectures is a cognitive activity of which only learners who are functioning at least at van Hiele level 3 or 4 are capable (see Burger & Shaughnessy, 1986). This partly explains why even the few learners, who successfully constructed/drew the required shape, noting all its essential properties, were still not able to state their observations as a conjecture.

Table 2 also reveals that more SAS learners were successful with the investigations than learners from the NS subsample. Of the 15 (75%) SAS learners who were successful with the first activity of investigation 1, 14 (70%) of them were also able to generalize their observation as a conjecture. Although conjecturing was generally difficult for the SAS learners (as it was for their NS counterparts), Table 2 indicates that these learners were more successful with investigations that dealt with the arc-angle properties of a circle (investigations 5 through 8) than those concerning the chord and tangent properties of a circle. As with their NS counterparts, the SAS learners were least successful with investigations that dealt with the tangent properties of a circle (investigations 9 & 10).

Like their NS counterparts, the grade 12 learners from SAS did not necessarily formulate their conjectures in formal technical statements even though many of them presented their conjectures in general terms. Siyabulela, for example, having established, through construction, that the perpendicular lines from the centre of a circle to two nonparallel congruent chords are equal (investigation 3), stated his conjecture in this way: “Two lines from circle centre perpendicular to equal chords are equal”. This may not be as technical as ***equal chords are equidistant from the centre of a circle***, yet his idea is clearly understandable. There were a few other learners in this group (SAS) who formulated their conjectures in general terms as Siyabulela did, demonstrating evidence of level 3 or 4 reasoning in the van Hiele hierarchy of the levels of geometric thought. The results as presented here find resonance with those of Siyepu (2005, pp. 64 & 65), in which in South Africa, “these grade 11 learners were unable to construct and measure angles … and nearly all of them could not generalize their observations as conjectures”.

Learners’ performance in the VHGT

Table 3 represents the results of the ***t***-test analysis of the percentage mean scores of the participating learners in the VHGT.

Table 3 Learners’ percentage mean scores in the VHGT

School	**N**	**Mean score**	**Std Dev.**	***t-value***	***df***	***p-value***

NS	23	37.61	11.27	- 1.88	45	0.067
SAS	24	44.79	14.63			

The t-test reveals that there was no significant difference between the mean score of the NS learners and that of the SAS learners even though the latter had a higher mean score than the former (t=-1.88, 45*df*, p>0.05). As with the CPGT, the results in Table 3 reflect a low level knowledge of geometry as the NS and SAS learners only managed to obtain mean scores of 37.61% and 44.79%, respectively in the VHGT. Given the nature of the test items that constituted the VHGT as a whole, these low mean scores could be interpreted to mean that the majority of the learners in both subsamples were at a low van Hiele level of geometric understanding.

The correlation between learners' scores in the VHGT and the CPGT

Although the analyses of the results of the learners in the CPGT furnish us with and extend our insight into participants' ability in formulating conjectures in circle geometry, its main purpose was to provide information on how learners' knowledge in this aspect of geometry relate to their van Hiele levels of geometric understanding. In order to provide an answer to the second research question, learners' scores in the CPGT were therefore correlated with their scores in the VHGT. Table 4 reflects the results.

Table 4 Correlation between learners' VHGT and CPGT scores

VHGT	**NS**	**SAS**
	CPGT	
	r = 0.32 p = 0.163	r = 0.55 p = 0.012

As Table 4 indicates, there was a moderate significant positive correlation between the VHGT and the CPGT scores for the SAS learners ($r=0.55$, $p<0.05$). For the NS learners, the correlation between their VHGT and CPGT scores is not significant ($p>0.05$), but nonetheless, it is positive. These results imply that for the majority of the learners in this study, success (or failure) in the VHGT also meant success (or failure) in the CPGT. That is, learners who had high (or low) scores in the VHGT had equally high (or low) scores in the CPGT.

To summarize, for the learners in this study, the van Hiele levels of geometric understanding correlate positively with ability to formulate conjectures in circle geometry. Therefore, raising learners' thought along the geometric thinking levels suggested by the van Hiele theory might prove useful in assisting them to ameliorate their difficulties with school geometry.

LIMITATIONS OF THE STUDY

The VHGT test used in this study, being a multiple-choice test, could not offer explanation or the reasoning behind learners' response choices, even when the test in its original design was meant to differentiate participants into discrete levels of geometric thought according to the van Hiele theory. Secondly, the analyses presented in this paper preclude an explicit interrogation of whether learners at different van Hiele levels also differ in their ability to formulate conjectures in circle geometry. While this study did not claim to have aimed at achieving this goal in its original design, insight into whether or not learners at different van Hiele geometric thinking levels show differential abilities in formulating geometric conjectures will probably be more useful to mathematics educators than the knowledge that the levels in general are positively related to conjecturing abilities the way that this study did.

CONCLUSION

This study employs the van Hiele model of geometric proficiency to interrogate selected grade 12 learners' ability to formulate conjectures in circle geometry in two African countries. Using the VHGT as a standard measure of learners' levels of geometric reasoning (Usiskin, 1982), this study explores the relationship between participants' van Hiele levels and their ability to formulate conjectures in school geometry. The results indicated that the van Hiele levels are positively related to learners' ability to formulate conjectures in circle geometry.

RECOMMENDATION

Given the significant positive correlation coefficient calculated between the SAS learners' scores in the VHGT and the CPGT ($r=0.55$, $p<0.05$) and the positive correlation coefficient obtained between the scores of the NS learners in these tests, it would seem reasonable to suggest that any effort aimed at enhancing learners' proficiency in stating conjectures in geometry should consider beginning with raising learners' reasoning in line with the van Hiele thought levels in geometry.

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SCAFFOLDING THE TEACHING AND LEARNING OF MATHEMATICS

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The purpose of this paper was to explore the value of scaffolding in mathematics classrooms. The paper was part of a larger study which explored how Master mathematics teachers used visuals as tools in mathematics classrooms. Activity theory was used as a framework to locate the study. Each activity system was interrogated within an interpretive paradigm. Data was collected from a convenience sample of six Master teachers by observations during teaching, video recordings of mathematics lessons and semi-structured interviews. Scaffolding techniques were used to support the learner's development of mathematical thinking. The contribution that the study makes is the knowledge that these scaffolding techniques and strategies may be used in any classroom within any social milieu.

INTRODUCTION

Having worked as a teacher and researcher within the mathematics field for the past 15 years I have been intrigued by how some teachers manage to assist their learners in grasping concepts in mathematics, with ease, as compared to other teachers. It was with this in mind that I embarked on exploring Master teachers use of visuals as tools in mathematics classrooms.

Based on anecdotal experience, many successful teachers use visuals as tools to enhance the teaching and learning of mathematics. Visual learning is an approach to helping learners communicate through imagery (Chmela-Jones, Buys, & Gaede, 2007). McLoughlin (1997) purported that visualisation enhanced a learners' ability to learn mathematics because it becomes a powerful cognitive tool. A key finding of the study was that visualisation played an important role in the mathematics classroom. Within the ambits of the study, visualisation means the ability to form and negotiate a mental image necessary for problem solving in mathematics. While mathematics encompasses many abstract notions, with visualisation, these abstract notions were made more accessible to the learners within the study. In the study the Master teachers used visuals as tools to assist with their teaching of concepts and ideas in mathematics; they turned to these visual tools because these tools acted as a scaffold to the teaching and learning of mathematics.

MASTER TEACHERS

The study explored how Master mathematics teachers teach, with the ultimate aim of using these findings to assist in improving mathematics instruction through innovations that may support teachers. In 2006 the Department of Education (KZN) announced that in terms of teacher development, 120 Master teachers would be appointed and an additional 2400 Master teachers would be trained (Makapela, 2007). This announcement was made due to the realisation that many schools in South Africa lacked qualified mathematics and science teachers. In South Africa, Master teachers serve the same purpose as a mentor or expert teacher. They are senior teachers with the potential to mentor new teachers.

While some of these criteria were used by the Department of Education to identify the Master teachers, the Master teachers in the study also exhibited determination and a commitment to improving the teaching and learning of mathematics within their classrooms. The Master teachers in the study all taught at Dinaledi schools. Dinaledi schools are schools that were selected by the National Department of Education. Dinaledi schools were intended to increase the participation and performance of Black learners and female learners in mathematics and science. The Dinaledi project which was established in 2001 was intended as a short term project providing teaching and learning resources to a limited number of schools.

VISUALISATION IN MATHEMATICS

In the 1990s Davis and Maher (1997) and Presmeg (1993, 1995, 1997) have conducted extensive research on why learners use visualisation in mathematics. More recently researchers (Arcavi, 2003; Diezmann & English, 2001; McLeay, 2006; Verstraelen, 2005) articulated many different reasons and benefits for how and why learners use visualisation in mathematics. Among these reasons is the perception that visualisation is an act in which an individual establishes a strong connection between an internal construct and something external.

Along similar lines, Nakin (2003) purported that visualisation is the cornerstone in the learning of mathematics in that mathematics depends on what is being visualised and the spatial abilities embedded in such visualisation. However, while it is an important skill, this skill receives little attention in the school curriculum. Visualisation entails the ability to form and negotiate a mental image necessary for problem solving in mathematics. Aristotle, as cited in Zazkis, Dautermann, and Dubinsky (1996, p. 437), believed that one could not think without having an image in one's mind. In a mathematics classroom, by allowing learners the opportunity to voice what they see in their minds promotes active discussion and interpretation. This enables others within the community of practice, to see what the learner sees, thus allowing for active communication. This assists in making the mathematics being taught in the classroom more accessible.

THE USE OF VISUALS AS TOOLS IN MATHEMATICS CLASSES

In the study, the Master teachers used visuals as tools for effective teaching and learning. Visual tools are typically conceived as instruments necessary for dealing with the ‘concrete real world’ rather than the ‘abstract world’ of symbolism. Anecdotal experience indicates that mathematics teachers often use visual tools with the intention of assisting learners to grasp a concept or problem in order to improve mathematical conceptual knowledge.

Recent research (Naidoo & Bansilal, 2010a, 2010b) indicated that in order for mathematics teachers to promote the development of visualisation skills in their learners, the teachers themselves need to be aware of the development of one’s visualisation abilities. Moreover, the presence of visual elements in today’s teaching and learning materials has increased with the integration of images and visual presentations within text (Branton, 1999). However, even though visual education is important for learners to successfully interact with shapes (Freudenthal, 1971) visual education is an ignored area of the national curriculum (Hershkowitz, Parzysz, & Van Dor-Molen, 1996; Naidoo, 2006). Thus the focus of the study was on the use of visuals as tools in the mathematics classroom. Based on anecdotal experience teachers often use visual representations unknowingly in class, for example, when they resort to the use of gestures, graphs, shapes, lines and diagrams. A gesture is any physical body movement (Maschietto & Bartolini Bussi, 2009; Roth & Lawless, 2002) that assists in a communication function (Sfard, 2009). Gestures include beats, deictic, iconic and metaphoric gestures.

The literature on Master teachers’ use of visuals reveals that this is a relatively unexamined phenomenon in South African classrooms. This perspective, namely, on teaching and learning through the use of visuals as tools in mathematics classrooms, recasts the relationship between what teachers teach and how they teach. It foregrounds the fact that the ways in which teachers teach and the ways in which learners learn are inextricable parts of the classroom culture. The study draws attention to the dilemma faced by many teachers in South Africa. Teachers are compelled to teach the same curriculum within the same timeframe nationally, regardless of the inequitable distribution of both human and material resources.

THEORETICAL FRAMEWORK

Activity theory was the theoretical framework used to locate the study. Each Master teacher’s classroom was considered as an activity system. Each activity system was diverse with respect to the social milieu within which it was situated. Activity theory assimilates the ideas of planning, negotiation, history and cooperation with the intention of understanding that consciousness and activity are interrelated and integrated (Nardi, 1996; Uden, 2007). Activity theory consists of a set of rudimentary principles that may be used as the basis for more specific theories. These principles include: The hierarchical structure of an activity, object – orientatedness, externalisation and internalisation, mediation and development. These principles are regarded as an integrated scheme. Each principle cannot exist in isolation. While there are three generations of activity systems, I use Engeström's (1987, 1993, 1994, 2001), second generation activity theory model in the study. This model was more proficient at illuminating shared activities and collaborative work.

METHOD

To explore the broader topic of Master teachers' use of visuals as tools in mathematics classrooms, three questions were generated to assist in this exploration. The first question identified what visuals Master teachers used as tools in mathematics classrooms. The second question explored how Master teachers used visuals as tools in mathematics classrooms. The third question interrogated why Master teachers used visuals as tools in mathematics classrooms.

The research represents a qualitative interpretation of Master teachers' use of visuals within mathematics classrooms. My intent was to focus on the Master teacher and to understand, interpret and explore the Master teacher's use of visuals in the mathematics classroom. The Master teachers in the study were probed with respect to their actions within the classroom as well as their responses on the questionnaire, to assist in gaining an in-depth understanding about their use of visuals.

The study involved the use of a questionnaire and interviews as a means of acquiring information. I interviewed six Master teachers from six Dinaledi schools in KwaZulu-Natal. Focus group interviews with learners from the six schools were also conducted. These focus group interviews served as a means of triangulating the data collected. For practicality purposes, I also used observations of lessons to assist in deriving more information about the use of visuals in the mathematics classrooms. This paper is based on one aspect of the broader study.

FINDINGS

Different visuals were used as tools in the mathematics classroom. These tools are displayed in Table 1[7].

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[7] Names of schools and teachers are pseudonyms.

<table>
<tr><th></th><th>Alan[7]
Orchid Secondary School</th><th>Karyn
Rose Secondary School</th><th>Dean
Daisy Secondary School</th><th>Maggie
Lilly Secondar y School</th><th>Penny
Tulip Secondary School</th><th>Sam
Carnation Secondary School</th></tr>
<tr><td>What visuals do Master teachers use as tools within mathematics classrooms?</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
Blackboard
Mathematics manipulatives
OHP</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
White board
Smart board
OHP</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
Blackboard
Mathematics manipulatives
Highlighters</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
White board
Smart board
OHP</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
Blackboard
Mathematics manipulatives
Pictures
Charts</td><td>Gestures
Colour
Diagrams
Signs, shapes, symbols and lines
Blackboard
Mathematics manipulatives</td></tr>
</table>

Table 1: Visual tools used by each Master teacher in the study.

This paper focuses on how the Master teachers used visuals as tools in the mathematics classroom. Based on the study I have found that rather than using direct teaching strategies or the traditional approach to teaching mathematics, all the Master teachers incorporated scaffolding techniques to support their learners' development in mathematics.

THE USE OF SCAFFOLDING TECHNIQUES

Scaffolding, which was introduced as an educational concept in the field of psychology by Wood, Bruner and Ross in 1976, describes the support given by a more expert individual during interactions (Sherin, Reiser, & Edelson, 2004).The use of scaffolding techniques necessitates that teachers supply their learners with tools that are necessary for learning. In the mathematics classroom, these tools could include diagrams, pictures, technology, and mathematics formula. According to Anghileri (2007), there are three levels of scaffolding.

Level 1: Examining the learning environment

Level 1 scaffolding relates to the manner in which the teacher organises the mathematics classroom. This organisation includes the use of artefacts. Artefacts embrace pictures and charts used in the form of wall displays. Environmental provision includes the prompts and incentives that exist within the classroom context (Siemon & Virgona, 2003). In the study the majority of the Master teachers (4 out of 6) utilised pictures, mathematics diagrams and charts as wall displays. The displays depicted mathematics theorems and proofs and were displayed prominently on the walls of the classroom.

Level 2: Exploring the teacher-learner interaction

Scaffolding at Level 2 includes different levels of teacher-learner interaction. This type of interaction relies on teachers' reviewing and restructuring what is happening within the classroom. During the reviewing process, learners ought to be encouraged to verbalise what they see and think. The learners need to be motivated to explain and justify their actions and comments. Through interpretation of learner comments, prompting and asking probing questions, teachers have a higher probability of identifying misconceptions and misunderstandings in mathematics thinking and learning. This leads to parallel modelling whereby, based on the teacher's identification of learner misconceptions and learner misunderstandings; the teacher creates and collaboratively solves tasks that share similar characteristics to the learners' problem.

In the restructuring of tasks, the teacher simplifies the problem or rephrases the learners' comments with the aim of negotiating meanings and taking the understanding forward. In the study Karyn used parallel modelling (Anghileri, 2007) when she used examples similar to the ones her learners had a problem with. She did this when she reviewed, restructured and worked on the solution until the problem solving process made sense to her learners. She shaped problems and examples stemming from her learners' comments. She supported her learners' understanding of tasks by operating from their ideas.

Dean also used parallel modelling to teach rotation of points about the y axis. He created and demonstrated the problem solving process. The problem he was working on had similar characteristics to the problems the learners in his class could not work with on their own. The learners could not rotate points 90 degrees about the y axis. Once he had completed his demonstration, Dean prompted his learners to try to solve the initial task. This was done with great success. Thus with his method of reviewing and restructuring the problem together with his method of parallel modelling, Dean was able to effectively scaffold the teaching and learning of rotation within his classroom. He made the mathematical rule of rotation visible to his learners so as to improve his learners' understanding (Karadag & McDougall, 2009) of this concept.

The use of representational tools

Level 3 scaffolding refers to the use of representational tools with the aim of generating conceptual discourse within the learner (Verenikina & Chinnappan, 2006). This includes the use of concrete materials to make mathematics more accessible to their learners. Sam used examples of the bricks on the wall to demonstrate what a number pattern was. He believed that if they (the learners) see the patterns on the walls, they could then associate it with the mathematical concept. Karyn drew different coloured shapes (rings and blocks) around specific steps in a solution process or around a particular formula. She used these symbols as signifiers within the lesson. During her lesson on number patterns and analytical geometry, Karyn used coloured blocks and rings to signify the key formulas learners ought to use whilst engaged in problem solving. Key terminology was also highlighted using these (rings and blocks) symbols.

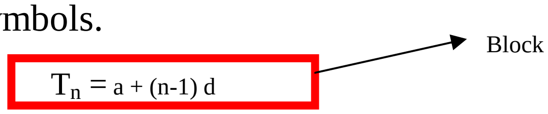

Figure 1: An example of a block used as a visual tool during one of the observed lessons

Maggie used her interactive smart board as one of her visual tools. She used her smart board to introduce mathematics manipulatives. For example, she used diagrams of triangles during the observed lesson on analytical geometry. She manipulated the size of these diagrams whilst discussing the concept of area. Based on experience, it is faster, more accurate and easier to demonstrate changes in height and angle measurements using virtual manipulatives (manipulatives using the computer) than it is to teach using the traditional ‘chalk and talk’ method. The time saved can be spent more profitably on probing learner comments and initiating interactive discussions. It is in this manner that the use of virtual manipulatives advances scaffolding for problem solving (Clements & McMillen, 1996).

As a concrete visual tool, Alan used a sheet of paper. He folded the paper along different lines of reflection (See Figure 2 that follows).

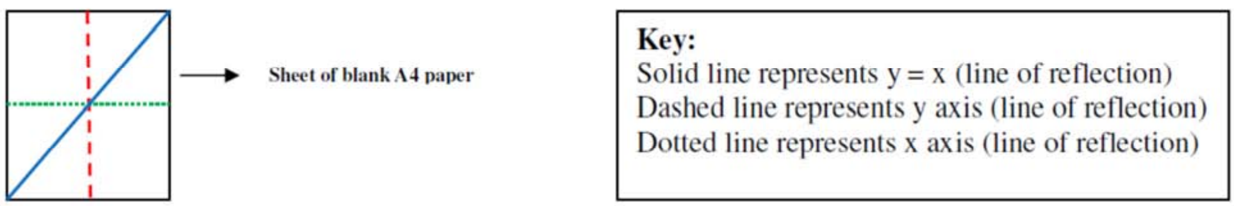

Figure 2: Paper folding used by Alan to demonstrate different lines of reflection

He then used deictic gestures to indicate positions of different points as they went through various transformations. The different coordinates were discussed and learners started developing their own rules for transformations about different lines of reflections. It was in this way that this piece of paper was used as a mathematic manipulative. The learners were thus able to grasp the concept of reflection, rotation and translation with the help of this concrete manipulative. This is evidence of how good manipulatives assist learners in shaping, reinforcing and linking different representations of mathematical ideas (Clements, 1999).

Dean constructed a stick with coloured rubber bands and used this visual tool within his activity system to demonstrate reflections about the x and y axis.

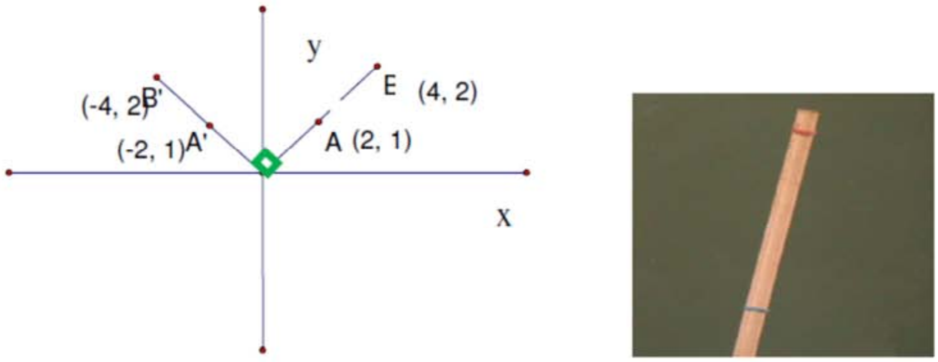

Figure 3: Dean's use of concrete sense making tools (The stick with the rubber bands)

Dean could have chosen to use the traditional approach of teaching by talking his way through the reflection exercise. But using the experience gained from over 20 years of teaching mathematics, Dean realised that learners needed to see something concrete to make the mathematics meaningful. Dean believes that using concrete manipulatives, makes the mathematics concepts and ideas easier to see. He stated that *"...by showing them* (the learners) *the rotation using the stick ... can see exactly how the position changes and I wanted them to see that initially ..."*

While scaffolding has become useful for teachers (Verenikina & Chinnappan, 2006), the purpose of scaffolding is to provide learners with teacher-supported transition. This implies that from looking at and listening to the teacher illustrate a particular mathematics concept, learners are now required to perform the skill independently. However, while the use of scaffolding in mathematics is necessary, scaffolding is useless on its own. It is necessary that scaffolding be complemented by mathematical understanding, together with the ability to think, perceive and analyse mathematically (Lewis, 2010).

DISCUSSION

Mathematics is regarded by many as a complicated subject. The extent of this complexity envelops both the learning and teaching of mathematics. This complication presents a stumbling block in society since success in South Africa is generally measured by the level of mathematics you know. Thus, it is important to be well equipped with the knowledge of mathematics. To substantiate this point, in order to gain access to higher paying occupations, learners are required to attain a high pass rate in their Grade 12 mathematics examination.

Many teachers are scarred by the consequence of being educated during the apartheid era. However in the midst of all the politics and bureaucracy some teachers are successful in making a difference in the mathematics classrooms. This success in some instances is difficult to explain and warrants exploration. It was as a result of these success stories that the study emerged. What was prevalent among these success stories was the use of visuals. Visuals were used as tools within the mathematics classroom to allow all learners access to the mathematics being taught. In some cases the Master teacher in the study used visuals to assist in the communication of mathematical ideas when the language of instruction was not the learners first. In many instances it served the same purpose as code-switching[8]. However instead of only using language to code-switch the Master teachers used visuals and language. In other cases, the Master teacher used the visual to clarify concepts being taught when language was not the issue.

Each Master teacher used their mathematics lessons to make a difference in their learners' lives. They were motivated and determined to make a difference. Visuals were used when text books, technology and other resources were not available. Master teachers were more interested in the solution process rather than focussing on attaining the correct answers. Visuals were also used to remove the abstractness of mathematics.

In light of the research done in the study, the contribution that the study makes is the knowledge that these techniques and strategies may be used in any classroom within any social milieu. Apart from poorly resourced schools these techniques may also be used in schools were the behaviour of learners proves to be the biggest obstacle to learning. These tools may also be used in classrooms where learners are not streamed into ability levels as is the case of the majority of schools in South Africa. These tools also proved to be highly effective in large classrooms. Additionally the Master teachers demonstrated the usefulness of their visual strategies in large classrooms with limited resources. The Master teachers in the study demonstrated that anything is possible provided that the teacher is determined and committed to making a difference in mathematics education. The ongoing professional development of the Master teachers also proved to impact positively on how each Master teacher taught in the classroom.

Essentially what was apparent in all the classrooms was that the Master teacher was not the distributor of knowledge, but rather each teacher acted as a guide for the educational experience of their learners. The role of the Master teacher in these classrooms was to help learners identify associations between their unprompted, everyday concepts and the formal concepts of the mathematics discipline. In this way learners played a more active role in their own learning and this led to intrinsic

[8] Code-switching is the concurrent use of more than one language. It means switching back and forth between two or more languages in the course of a conversation.

motivation. The learners' enjoyment, fulfilment and interests were emphasised. It was through these observations that the classrooms in the study were seen as progressive learning spaces rather than traditional teaching spaces. While these researcher thoughts are based on what was observed, each Master teacher also provided reasons for their use of each visual.

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CONTEXT-BASED PROBLEM SOLVING INSTRUCTION TO INDUCE HIGH SCHOOL LEARNERS' PROBLEM SOLVING SKILLS

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This study explored the use of a context-based problem solving instruction to develop the 10th grade mathematics learners' problem solving skills. During a two-week-intervention-programme learners were given worked-out examples to facilitate acquisition of problem solving skills. A convenient sample (n = 57) of learners participated in the study. The pre- and post- achievement tests were written to examine the influence of new instruction on learners' performance. Using a Spearman's Brown formula the reliability of the test was estimated at r = 0,92. Results showed that the learners' problem solving skills improved (p < 0,05). Cognitive Load Theory is used to explain and analyze the cognitive aspects that are said to have influenced learners' cognitive processes during solving problems.

INTRODUCTION AND BACKGROUND

In South Africa, the new National Curriculum Statement (NCS), which is based on the principle of Outcomes-Based Education (OBE), positions problem solving as one of the critical skills in mathematics (Department of Education (DoE), 2006). For instance, the curriculum statement for mathematics emphasizes that "mathematical problem solving enables us to understand the world and make use of that understanding in our daily lives" (DoE, 2006: 20). However, emphasis on learning about problem solving in mathematics is not evident in learners' outcomes. This is evidenced by the matriculation results which highlight serious problems in mathematics instruction. For instance, prior to 2008, the top 11% of schools accounted for 71% of higher grade mathematics passes, and the bottom 81% of schools produced only 16% of higher grade mathematics passes, an average of 1% per school (Gauteng[9] Department of Education (GDE), 2010: 10). These results imply that a majority of our learners cannot analyze problem information, and are unable to utilize problem solving skills to come up with a strategy to find envisaged solutions.

From this background I think there is a need to find alternative instruction that promotes the acquisition of problem solving skills in mathematics. To achieve this I have established whether incorporating the use of real-life contexts in the mathematics problem solving processes might foster deeper and meaningful understanding, and also enhance problem solving skills by the learners. The

[9] Gauteng is one of the nine provinces of South Africa.

mathematics curriculum document emphasizes that “tasks and activities should be placed within a broad context, ranging from the personal, home, school, business, community, local and global” (DoE, 2006: 19). To this end I argue that mathematics tasks situated within learners’ realistic contexts can promote learning and, as a consequence, improve learners’ problem solving skills.

I have focused on the use of a *context-based problem solving instruction* to develop learners’ problem solving skills in mathematics. In this paper, a *context-based problem solving instruction* refers to a teaching approach in which everyday problem solving knowledge and practices are uncovered when learners are exposed to tasks giving meaning to their everyday experiences. Hence, the objective of this study was to investigate the effect of implementing a context-based problem solving instruction on learners’ performance in the area of mathematical problem-solving. To this end, I have asked:

- *To what extent can the incorporation of a context-based problem solving strategy influence learners’ performance in mathematics?*

MOVING TOWARDS A CONTEXT-BASED PROBLEM SOLVING INSTRUCTION

There is a common agreement in mathematics education about the significance of incorporating real-life contexts in mathematics instruction. Several studies have reported benefits in the integration of real-life contexts with problem solving in mathematics. Fuchs, Fuchs, Finelli, Courey, and Hamlett (2004) conducted a study to investigate the possibility of knowledge transfer to learners when real-life problems were used during instruction**.** This study involved grade 3 learners who were given four different types of real-life problems to solve. Results showed that knowledge transfer was a challenge in that learners needed to develop relevant schemas for recognizing novel problems as belonging to familiar problem types.

Khumalo (2010) conducted a qualitative case study to explore a grade 8 natural science teacher who integrated learners’ real-life contexts in developing the curriculum, during instruction, and assessment. Learners made use of their real-life contexts to negotiate meaning during problem solving. They discussed context-based activities in groups and solved problems in their context, produced the portfolio boards by using the resources from their context and presented their portfolio boards to class (Khumalo, 2010). Findings revealed that “problem solving needs more time but maximizes non-routine thinking.” (Khumalo, 2010: iii).

In 2005, Carlson and Bloom conducted an experiment in which they presented 12 university research mathematicians with 5 complex real-life problems (each individual answering 4 of them) that required knowledge of foundational content and concepts such as basic algebra or geometry. Carlson and Bloom (2005) concluded that the participants' abilities to play out possible solution paths to explore the viability of different approaches had contributed significantly to mathematicians' efficient and effective decision making processes. In this study, prior knowledge of the subject served as a relevant context that facilitated problem solving in novel situations. These studies help us to realize that:

(1) *there is a relation between the extent of problem schema acquisition and success rate in problem solving* (Fuchs, et. al., 2004);
(2) *familiarity plays a role in the development and acquisition of problem solving skills* (Khumalo, 2010); and,
(3) *when real-life contexts pose as prior-knowledge problem solving skills is facilitated* (Carlson, et. al., 2005).

Given that both international and national studies reveal poor mathematics skills for South African learners, there is a need to conduct similar studies in a South African context (Bansilal, James, and Naidoo, 2010).

THEORETICAL FRAMEWORK

This study uses the *worked-out example effect* which is postulated within Sweller's Cognitive Load Theory (CLT) (Sweller, 1994). CLT provides a relevant framework for investigations into cognitive processes and instructional designs linked to problem solving activities. Worked-out examples *effect* is a set of problem-related examples that presents an instructional tool to teach problem solving skills. According to Moreno (2006: 170), working examples usually consist of modelling the process of problem solving in a well structured domain such as physics or mathematics by presenting an example problem and demonstrating the solution steps and final answer to the problem. In this model the teacher allows learners to study several examples before they solve problems on their own. Teacher demonstration, as a form of guided support, is an essential ingredient that prevents learners from pursuing unproductive paths.

Furthermore, this procedure is said to reduce the load imposed on the working memory during cognitive activity such as problem solving (Renkl and Atkinson, 2007). Given that working memory is a limited capacity processor it is important to observe that when this limit is exceeded learning processes is affected. Worked examples reduce problem solving demands by providing worked-out solutions. It promotes learning because more of the learners' limited working memory capacity can be devoted to understanding the domain principles and their application to the problem at hand (Renkl and Atkinson, 2007). With extended exposure to worked-out solutions problem solving schemas can be processed automatically, and according to Moreno (2006), this process requires minimal working memory resources. Using the

CLT, I argue that instructional designs that take advantage of the worked-out examples effect promote learning and facilitate acquisition of problem solving skills by novice problem solvers.

METHOD

Design

Though a mixed-approach design was used to investigate the influence of a context-based problem solving instruction on learners' problem solving skills, this paper only reports on data collected from the achievement test. The test was administered at pre- and post-stages.

Participants

The study involved grade 10 mathematics learners (n = 57) from one school that participated conveniently (Cohen, Manion, and Morrison, 2007). The mean age of the learners was 18.44 (*SD* = 0.75). The school was located in a township setting in the Gauteng province of South Africa. Generally, township schools are in areas that were demarcated for blacks during the apartheid[10] era and were not adequately resourced and provided for by the then government. Schools from this background are usually non-performing in mathematical terms (Van der Berg, 2007).

Instruments and measures

I developed the achievement test and also planned all the activities carried out during the intervention instruction. Problems were introduced to learners in a form of worksheets. The problem solving items prepared for the test were compatible with learners' levels in accordance with National Curriculum Statement (DoE, 2006). This means that the items were compiled in line with assessment guidelines provided in the departmental curriculum policy documents. Using the Spearman Brown prophecy formula, the reliability of the test was found to be 0,92.

The *content* and *face validity* of the test were achieved on the basis of collegial judgments from university professors and lectures, subject advisors from the local Department of Education District, and from mathematics Heads of Department and teachers. However, I also observed that "it is important to conduct an empirical assessment of the adequacy of those judgments" (Rubin and Babbie, 1993:67). To address this aspect the ***criterion-related validity*** was also established for the test. In this case, the criterion was whether the test would provide the feedback on the status of learners' problem solving skills. The validity of the test was finally determined by

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[10] Apartheid is a term used to describe a system of legal racial separation which dominated the Republic of South Africa from 1948 until 1993. In this system, white people were the most privileged section of the society. Blacks, on the other hand, were the most disadvantaged group.

its ability, on the basis of its scores, to distinguish between problem solving skills of learners.

Procedure

Learners were involved in a two-week enrichment period in which I introduced them to a context-based problem solving instruction. The study began by giving each participating learner a familiarization problem solving task prior to other different types of activities. This prepared them for pre-test. Open-ended problems such as the one below were designed and given to learners:

> *"Thembi spent R475 on two skirts and a pair of shoes. How much did she pay for the skirts and the shoes?"*

Although this problem appeared not to provide sufficient information for problem solution, it was however perceived capable of promoting mathematical reasoning and thinking, and was prone to induce learners' problem solving strategies. At the end of pre-test learners' scripts were collected and labelled using codes for identification purposes. The context-based problem solving instruction was then introduced to learners. Before the lesson, learners were arranged into groups of 5 – 6 learners in which learners discussed the problem and possible strategies for solution. At the beginning of each lesson I introduced the topic to be discussed. This was followed with the distribution of a worksheet with 3 to 4 worked out examples with full steps on how to solve the problem. Examples were followed with problems such as the ones below.

> ***Example 1*** *Sipho sees a television for sale and wants to buy it. He has no money so he decides to borrow money from his sister. His sister agrees to lend him R500, but says that she will charge him 10% of the original amount every month until the money is paid back. Sipho can only pay the money 6 months later. Let's have a look at how much Sipho will owe his sister.*

Now	***1 month***	***2 month***	***3 month***	***4 month***	***5 month***	***6 month***
R500	*R550*	*R600*				

Questions:

1. *Copy and complete the table above;*
2. *Why do you think that R50 is being added on each month?;*
3. *How much would Sipho owe if he paid the money back after 9 months?*

> ***Example 2*** *Discuss and calculate the difference in the price of a bicycle that costs R795 in cash or R200 deposit and R165 per month over 6 months.*

Most of the items in worksheets were drawn from the learners' textbook (activity book). Learners were requested to discuss examples amongst themselves. While learners were busy with the problems in worksheets I monitored them and asked questions when necessary, such as:

What came into your mind when you were first confronted with this problem?; Do you understand this problem?;
What is your solution strategy for this problem?; etc.

I asked these questions to arouse learners' opinions about the new instruction, and also to encourage them to externalize their problem solving thoughts and strategies. I also encouraged them to think aloud about their problem solving tasks, to discuss and explain their steps to others, and also to approach problems from different perspectives. I requested them to provide complete solutions for all their work.

Results

The level of problem solving skill acquisition was measured by the performance in the achievement test. The pre-test mean score was 18.54 (SD = 6,827; n = 57) while the post-test mean score was 21.35 (SD = 7,328; n = 57). I observed that the mean scores, particularly those of the pre-test, were low. Form the pre-test results I was able to assume that learners were in their early stages of problem solving skill acquisition. Having observed the increase of performance from the pre-test performance to post-test performance I needed to verify that this improvement was due to problem solving intervention. To determine the effectiveness of the new instruction, the mean scores of the pre- and post-tests were compared using a t-test at the significance level of 0,05. The results of the t-test analysis are presented in table 1 below.

Table 1: Statistical results of the t- test analysis for the achievement test.

Test	group	n	$\bar{x}$	SD	SEM	t	p-value
Pre-test	grade 10 learners	57	18.54	6.827	0,90	2,116	0,0366
Post-test	grade 10 learners	57	21.35	7,328	0,97		

*Significant at 0,05 level

The results above suggest that the performance of the learners in a problem solving achievement test has improved significantly ($p < 0,05$). To this end, it is possible to conclude that the context-based problem solving instruction designed to improve learners' problem solving skills has been effective. In this regard, the research question, which asked*: To what extent can the incorporation of a context-based problem solving strategy influence learners' performance in mathematics?* has been also been answered.

DISCUSSION AND CONCLUSIONS

In this study, I sought to explore the effectiveness of a context-based problem solving instruction to the 10th grade mathematics learners. With this objective in mind, I formulated the problem of the study as follows: *"To what extent can the incorporation of a context-based problem solving strategy influence learners' performance in mathematics?"*. The findings suggested that there was a significant difference in learners' problem solving performance before and after intervention. Furthermore, CLT helps us to understand how this difference came about. From the CLT perspective, we learn that "relieving students from any problem solving demands in the first place allows them to direct their full processing capacities at developing a basic understanding of the domain principles before any attempts to solve problems"(Schwonke, Renkl, Krieg, Wittwer, Aleven, and Salden, 2009). So it is possible to use CLT to argue that the load usually associated with problem solving can be reduced when learners are introduced to worked-out examples at the initial stages of problem solving skill acquisition. In terms of CLT, unsupported problem solving can pose heavy cognitive load on the learners because it is characterized by errors and unproductive search procedures (Schwonke, et. al., 2009). In this study I have used worked-out examples, designed from learners' known real-life contexts, to reduce cognitive load. We have observed that worked-out examples familiarized learners to problem solving steps. It also provided a bridge for learners to navigate from the known to the unknown, and as a consequence, develop effective problem solving skills.

From the results of this study I am able to conclude that the context-based problem solving instruction introduced in this study can positively influence learners' performance in mathematics. This conclusion supports previous studies that it is possible to improve learners' problem solving skills through instruction (Khumalo, 2010; Carlson and Bloom, 2005; Fuchs, et. al., 2004; `etc.). In all these studies instruction was used to improve learners' problem solving skills in mathematics. I conclude by highlighting a need to socialize teachers to effective instructional designs for problem solving. I also suggest additional research to explore further the correlation between problem solving performance and meta-cognitive skills.

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MULTIPLE REPRESENTATIONS OF GRADE 12 ROTATION, REFLECTION AND MATRIX MULTIPLICATION

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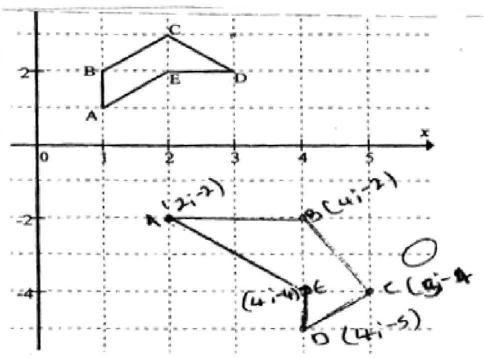
This paper is a response to Grade 12 learners' misconceptions on the concepts of reflection and rotation. Three representations of pictorial, verbal and symbolic are discussed in detail on the concept of reflection in the y-axis and deduced on the concept of rotation. Though matrix multiplication is not in the Grade 12 curriculum it is used as a powerful concept to generate and generalize standard transformation procedures. The merits of multiple representations are discussed using the examples provided. Bearing in mind that teachers need both procedural and conceptual mathematical knowledge in order for them to create effective learning environments, the paper theorizes the rationale of teaching these transformation concepts. It is hoped that teachers and learners can get instructional and learning insight from reading the paper.

INTRODUCTION

There is generally consensus among educators that learners can understand mathematical concepts when presented in multiple ways (Gagatsis *et al,* 2004). Multiple representations can be viewed as external mathematical embodiments of ideas and concepts that can provide the same information in more than one form (Özgün-Koca, 1998). Multiple representations can also be viewed as ways or forms that symbolize, describe and refer to the same mathematical concept. Bransford, Brown and Cocking's (2000) theory of multiple intelligence states that there are many ways in which people can learn new concepts using different types of media and knowledge that can show integration of different parts of mathematical sections. In cases where some concepts are not included in the learners' curriculum, teachers may still use the methods to verify results or as extension to challenge learners with high aptitudes. For instance, matrices are very useful in solving simultaneous linear equations and in generalizing transformation geometry concepts that although the concept is not in the grade 12 curriculum teachers can use them to demonstrate multiple solution strategies to enable learners to make choices from a variety of methods. Nonetheless, there is no right representation that can be described as the best because the effectiveness of each one of them depends on a student's learning style and the level of understanding that a representation affords a learner. Intellectually challenging learning environments can be created in classrooms where learners can seriously engage in thinking using multiple representations to express mathematical concepts and their understanding of them. Concepts on transformation geometry such as rotation, reflection and translation are encountered by learners in their environments on a daily basis yet some of the learners find them difficult to comprehend in mathematics classrooms. An example of learners' misconceptions on reflection about the origin in the *x*-axis is shown in Figure 1 below.

Figure 1: A Grade 12 learner's representation of reflection of a polygon in the *x*-axis



Source: National Senior Certificate Grade 12 examination for November 2008, item 3.2.

The above graphical representation of the learner's image does not show how the image of the original polygon comes about in order to understand the approach used. However, a common observation may be that the learner changed the signs of all the coordinates of the corners of the polygon in determining the coordinates of the image. Changing the signs will not have changed the shape and size of the polygon, making this observation not plausible. This may lead to another observation that the learner had problems on the concept of plotting coordinates. After all these possible observations and interpretations one can conclude that the learner does not show understanding of the basic principle that reflection does not change the shape of a polygon. Such misconceptions are common to some Grade 12 learners. In attempting to reduce such misconceptions, the present study attempts to share insight into the nature and possible contributions of multiple representations in the learning of the transformation geometry concepts of rotation, reflection and translation. These concepts were carefully chosen for their common character that a transformation under them conserves the original shape. It is hoped that the significance of this presentation lies in the anticipation that it may provide insight that may increase mathematics teachers' content and pedagogical knowledge about transformation geometry concepts and how to present them during teaching. Dynamic Technology such as Geometer Sketchpad or Geogebra can be used to verify transformations deduced and for in-depth analysis of general points on transformation trends when learners have understood the basic principles manually. This paper acknowledges the place of technology in the mathematics classrooms but encourages mastery of basic concepts before technology can be used in order to nurture the value of the division of labor between learner thinking and technology as aide for fast and accurate procedures (Brumbaugh & Rock (2001).

THEORETICAL BASIS OF MULTIPLE REPRESENTATIONS

In the course of learning students necessarily need to engage, observe and find out specific patterns or rules governing the concept under focus for them to conceptually understand it (Hwang, *et al* 2007). For the engagement, observation and pattern figuring to take place, the content they learn should be presented using multiple representations. Mathematical multiple representations entail presenting a concept on a table or grid, graph or diagram, formula or using symbols abstracting concept. Brunner's theory of instruction summarised these multiple representations into three learning stages of enactive, iconic and symbolic (Eugino, 2009). The enactive mode involves learners using some known aspects of reality and imagination to produce a pictorial representation of the concept without using words. The iconic mode involves internal imagery, where prior knowledge is used to form a set of symbols that can stand for the concept. Visual or other sensory associations are vital for perceptual organisation and techniques for economically transforming perceptions into meaning for an individual learner. The symbolic representation is based upon abstract thinking, discretionary and flexible thought that enables learners to present a concept in their own words or in the form of an algebraic formula that summarizes a concept.

Learners in mathematics classrooms possess a host of variables such as learning styles, prior knowledge, interests and motivations that makes teaching them a complex activity (Even, 2005). By presenting concepts in multiple ways learners with different learning styles can be catered for in the mathematics classrooms. Capel, Leask and Turner (2009) noted three types of learners, namely, visual, auditory and kinaesthetic learners. Visual learners learn through seeing things on diagrams or pictures. They understand a concept that is presented in pictorial form. Auditory learners understand concepts through explanations which they hear. They prefer to have things explained to them verbally rather than to read written information because they comprehend and process verbally presented information most effectively. The kinaesthetic or tactile learners learn through experiencing or doing things themselves. They use all their senses to engage in learning. As such they enjoy hands-on approaches to things and learn through trial and error. Mismatching teaching and learning styles has the potential to lead to learner disappointment, de-motivation, misunderstanding and underperformance (Visser, *et al*, 2006).

The active learning credo that may cater for learners' different learning styles in the mathematics classroom may be the one provided by Silberman (1996, p. 1).

What I hear, I *forget*.

What I hear and see, I *remember a little*.

What I hear, see, and ask questions about or discuss with someone else, I begin to *understand*.

What I hear, see, discuss, and do, I *acquire* knowledge and skill.

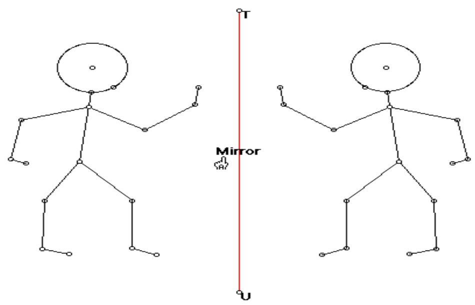
What I teach to another, I *master*.

This credo epitomizes the importance of multiple representations in the mathematics classroom to enable learners to use most of their senses during engagements with concepts they are learning. A learning environment that encourages listening, observations, discussions and explanations can be more ideal for learner conceptual understanding than their passive reception of mathematical content and procedures. An attempt is made in the rest of this paper to show how multiple representations can be used in the teaching and learning of some transformation geometry concepts.

MULTIPLE REPRESENTATIONS OF REFLECTION

Reflection is a common transformation that learners encounter on a daily basis using mirrors. Transformation involves an original shape that changes its position to form an image. In a reflection there is a mirror line and each point of a polygon is moved the same distance from the mirror line as an equivalent point on the image. Reflection can occur horizontally or vertically or in any line. The shape remains the same size after reflection irrespective of the number of mirror lines. Figure 2 shows a reflection of a model person.

Figure 2: Reflection of a model person through a mirror line

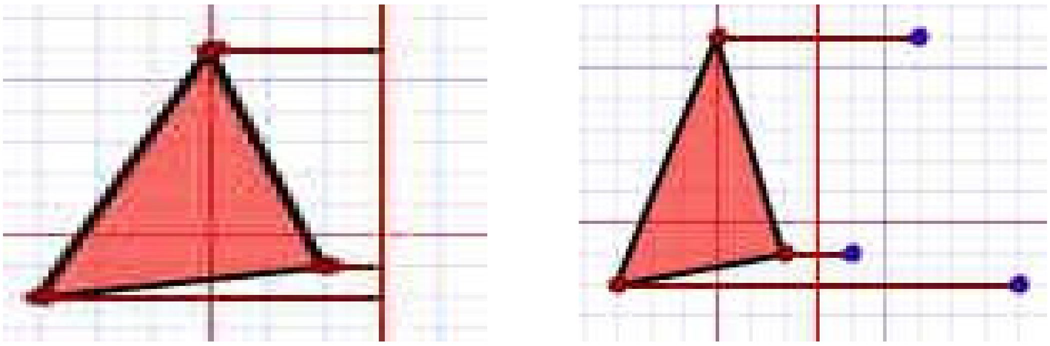


When the concept of reflection is made in a practical way like in Figure 2 using the medium of a mirror it can be further developed using a triangle in order to analyse the mathematical properties involved. Using different representations an attempt is made on how to develop the concept among learners.

Pictorial representation

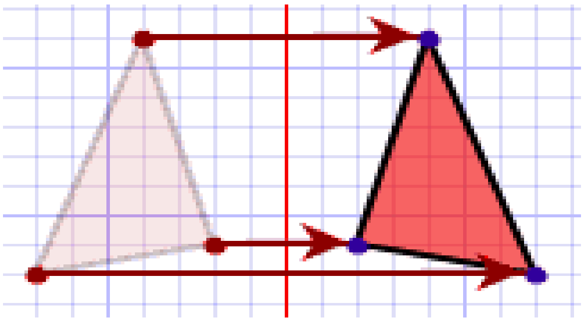
Pictorial representation involves learners using some known aspects of reflection without using words. For instance, learners can identify the mirror line as the *y*-axis and then use their imaginations to fix the corners of the image triangle before joining them as shown in Figure 3.

Figure 3: Process of reflecting a triangle in the y- axis.



After fixing the three corners of the image triangle, learners can join the corners using straight lines as shown in Figure 4.

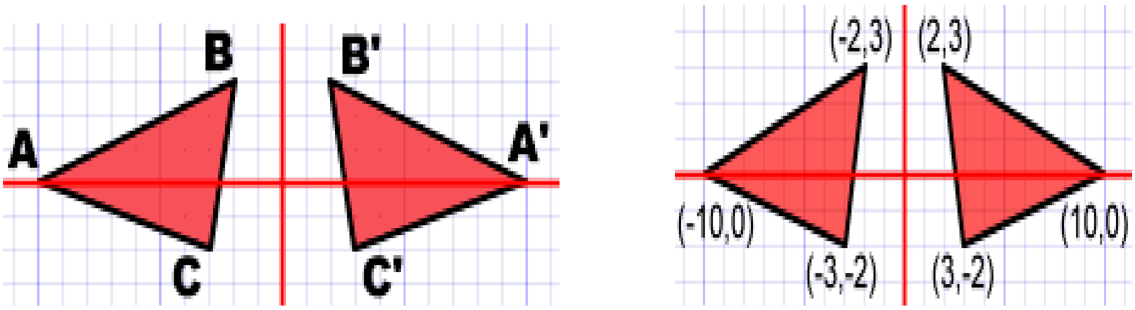
Figure 4: A triangle and its image after reflection in the y-axis



Verbal representation

In the second representation learners use their prior knowledge to form a set of symbols that can describe the concept of reflection in order to verbalise the processes involved. The use of coordinates can be useful in order to study the patterns occurring and analysing how the corners of the original triangle are changing in the image triangle for them to verbalise the process.

Figure 5: Searching for coordinates after reflection in the *y*-axis



A point-wise generalisation can emerge from noticing that $(a;b)\rightarrow$ Reflected in the y-axis $\rightarrow (-a; b)$. From this generalisation a pattern can be noticed to emerge that during reflection in the y-axis the coordinates of *x* change signs and those of *y* remain unchanged. The learners can verbalise this parsimonious conclusion and on their own make an intelligent guess and establish the case for reflection in the *x*-axis.

Symbolic representation

During the symbolic representation learners need to abstractly perform the process of reflection without using diagrams. Mere changing signs to determine the coordinates require verification in order to challenge fast learners. An algebraic analysis of the process involved may take the following argument:

Let the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ reflect the point (–2,3) and (–10,0) to (2,3) and (10,0) respectively in the y-axis. It is important to note that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} -10 \\ 0 \end{bmatrix} = \begin{bmatrix} -10a \\ -10c \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} \Rightarrow \begin{Bmatrix} -10a = 10 \Rightarrow a = -1 \\ -10c = 0 \Rightarrow c = 0 \end{Bmatrix}$$

And $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\Rightarrow$ –2a +3b = 2; substituting a by –1; 2 +3b =2; b = 0

Also –2c + 3d = 3 $\Rightarrow$ 0 + 3d = 3 and d = 1; since c = 0.

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

This means that to reflect a shape in the y-axis, multiply the coordinates of the shape by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Testing this assertion yields $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{vmatrix} -3 \\ -2 \end{vmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ which is the coordinate of the third corner of the rotated triangle in Figure 5. Using similar procedure learners can find a transformation matrix for reflection in the *x*- axis. The same process as the one give above can be repeated or fast learners can make an intelligent guess can deduce such a matrix to be $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Whichever way is used learners should be able to discuss and convince each other of why this intelligent guess makes logical sense.

Reflection in the line y = ±x can similarly be established and generalised as shown below.

Reflection about the line y = x swaps x and y, i.e. we have x' = y; y' = x. The matrix associated with reflection about this line is thus

$$M_{xy} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, reflection about the line y = –x swaps x and –y, i.e. we have x' = –y; y' = –x. The matrix associated with reflection about this line is thus

$$M_{-xy} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The concepts of rotation and translation can similarly be developed using the three representations. Due to space limitations the three processes of representations are not presented in detail and only some guidelines are provided. A rotation is a transformation that is performed by "spinning" the object around a fixed point known as the centre of rotation. A rotation can be in a clockwise or anti-clock-wise direction with a specified angle in degrees. The pictorial, verbal and symbolic representations follow those of the reflection transformation. The rotation transformation matrix about the origin through a right angle in a clock-wise direction that can be deduced using matrix multiplication is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and that of anticlockwise direction $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The inquisitive learners may be interested to find out the result of rotating a polygon through a general angle θ. This can be generalised as

$$R[\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

A summary of the above transformations for a point (a; b) is shown below:

Transformation	**Transformed point**	**Left factor matrix**
1. Reflection in y-axis	(–a; b)	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
2. Reflection in the x-axis	(a; –b)	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3. Reflection in y = x	(b; a)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
4. Rotation about origin through 90^0	(–b; a)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
5. Rotation about origin through -90^0	(b; –a)	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
6. Rotation about θ		$\begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix}$

Translation is the most basic transformation. Under a translation every point of the original image is moved the same distance and direction to form the image. A translation follows a rule of how many steps left or right and up and down. Steps to the right of the original polygon are perceived as positive and those to the left negative. In a similar way steps up are positive and those down are negative. The symbolic stage can lead to the generalisation $T(x, y) = (x \pm a, y \pm b)$, where a is the constant horizontal (x-axis) distance moved and b is the constant vertical (y – axis) distance moved.

DISCUSSION

When exposed to the three modes of representation of reflection as presented above, learners have freedom to choose which representation makes sense and is easy for them to perform. The learners with visual and kinaesthetic learning styles can use pictorial representation for them to be guided by diagrams to understand the processes that take place leading to the image diagram. The auditory learners may remember the changing of signs of the coordinates and apply them without showing working. The correct use of any of the three representations leads to a viable image. Using their preferred representation method the errors or alternative conception showed by the learner in Figure 1 may be reduced.

Mathematics teachers are strongly encouraged to use multiple representations during teaching in order to make learners make connections between different concepts during the complex process of mathematics learning in order for learners to better understand important mathematical concepts. For instance, during the pictorial representation phase learners can review and strengthen their psychomotor skills when they represent their intuitive ideas of reflection using a mirror. In the verbal phase they use the Cartesian plane to explore the changes in the coordinates of the sides of the triangle under reflection in order for them to determine the corresponding coordinates of the corners of the image. Finally in the symbolic phase they use algebraic reasoning, matrix multiplication and simultaneous linear equations to determine a general transformation matrix that reflects the triangle in the y-axis. Generalization of particular situations to more general situations is an important skill that learners should develop in class so that when they face a similar problem in future they can easily use their prior knowledge to solve it without going through intuitive steps. The connections between different mathematical concepts that multiple representations demand can enable learners to develop deep and more flexible understanding of the concept under focus. Other important uses of multiple representations in mathematics are that they enable multiple concretizations of a concept, they can be used to reduce some learning difficulties, they can make mathematics more attractive and interesting and can facilitate cognitive linking of representations (Keller &Hirsch, 1998).

The use of multiple representations can also be helpful to present clear and better pictures of concepts or ideas. This is possible because multiple representations can facilitate learning from familiar situations to abstract ones. For instance, the transition from pictorial representation to symbolic representation is facilitated by learners' prior knowledge of a mirror as a reflection agent. The use of the aid of a mirror helps learners to conceptualize the process of reflection. This early conceptualization is useful to derive a transformation matrix that can be used as a generalization for reflecting polygons abstractly without the physical aid of pictures or graph paper. The combination of images and texts that is evident in multiple representations has been associated with gains in learner understanding of mathematical concepts (Ainsworth & Loizou, 2003). Furthermore, audio representations when appropriately combined with images have been connected with increase in learner understanding and performance on comprehension of mathematical procedures.

Silberman's (1996) credo mentioned earlier has strong implications for reform teaching that encourages mathematics teachers to employ limited passive traditional methods and actively engage learners in their learning. This view was born out of the realisation that mathematical concepts are not fixed, are tentative and can be created from objects in the learners' environment. Mathematical knowledge can be socially constructed through reflective abstraction on observations and patterns. During the processes of construction there exist cognitive structures that are activated. These cognitive structures are continually under development. This makes purposive activities like those inherent in multiple representations induce transformation of existing structures to develop deep understanding of mathematical concepts. The tentative nature of mathematical knowledge is true as portrayed by a mirror that is common in learners' environments and is fundamental in the development of the concept of reflection.

The traditional passive methods of teaching mathematics have not been successful to make learners deeply understand mathematical concepts because they "hear and forget" teachers' demonstrations of mathematical algorithms (Silberman, 1996, p. 1). In such classrooms learners get bored and use the acronym TIRED (Tedium, Isolation, Rote Learning, Elitism and Depersonlisation) to describe school mathematics (Hagginson, 2008). Such l feelings about mathematics often make earners develop a fear of mathematics. After realising the underachievement of learners in classes dominated by traditional teaching methods reform pedagogy encourage learner autonomy informed by theories of constructivist instruction. In such classrooms learners are the focus of the learning process in a community of learning. They experiment with their ideas, use models from their environment to construct mathematical concepts, ask questions, explain their understanding and discuss concepts with their peers. As noted by Silberman, (1996, p. 1) learners understand what they "hear, see, discuss, do and teach others".

CONCLUSION

This paper attempted to provide insight on how to use multiple representations on the concepts of rotation, reflection and translation. The three representations of pictorial, verbal and symbolic that were discussed in detail to develop the concept of reflection in the *y*-axis are suitable to constructivist classrooms that are learner centered. In such classrooms learners can use their prior knowledge to develop new concepts, teachers scaffold knowledge gaps through asking open-ended question to enable learners remain focused on the concepts under focus. The multiple representations discussed are assumed to develop deep understanding of the concepts on transformation geometry. When they have understood the transformation concepts, learners can be exposed to dynamic technology such as Cabri-Geometry, The Geometer's Sketchpad, The Geometry Inventor and Geometry Expert only to mention a few to consolidate their understanding or verify their transformation images. Naturally it is not possible to exhaust all the techniques for developing learners' understanding in a limited space like a conference paper. More studies can explore teachers' and learners' current practices on transformation geometry in order to have broad insight on the nature and sources of learners' problems on the topic.

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ACE: MATHEMATICAL LITERACY QUALIFICATIONS – SOME INSIGHTS FROM KZN

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Higher Education Institutes in South Africa are contemplating recurriculation of Advanced Certificates in Education (ACE). This paper focuses on aspects of two ACE programmes that train in-service teachers to teach mathematical literacy in KwaZuluNatal. We look at the coherence and composition of course content in the light of content knowledge, domain pedagogy and general pedagogy and explore correlations between success rates of modules. The results of the preliminary study suggest that teachers' senior certificate marks for mathematics could be an indication of success in mathematical literacy studies despite the passage of intervening years. We believe that there could be some insights from past deliveries that could inform developers of further mathematical literacy programmes.

INTRODUCTION

Advanced Certificates of Education (ACE) programmes from all South African Higher Educational Institutions (HEIs) are currently being placed under the spotlight. According to the November 2010 Draft Policy on the Minimum Requirements for Teacher Education Qualifications selected from the Higher Education Qualifications Framework (HEQF), ACEs will have to be recurriculated into either Advanced Diplomas in Education (ADE) at National Qualifications Framework (NQF) level 7 or Advanced Certificates in Teaching (ACT) at NQF level 6 (Department of Higher Education and Training, 2010). The quandary HEIs are presented with is that the HEQF is under review and there is speculation as to how students could, in the future, negotiate an academic path vertically through the new qualifications towards Masters and Doctoral degrees.

Mathematical Literacy (ML) is one of the 'new kids on the block' in the National Curriculum Statement (NCS) and in order to reskill teachers to teach ML effectively, many HEIs developed and delivered ACE: Mathematical Literacy (ACE: ML) qualifications. This paper describes two such ACEs that have been delivered by two different universities in KwaZuluNatal (KZN) and begins to explore the differences and similarities between the two qualifications. This study is part of a larger NRF funded project in its preliminary stages and, as such, the results are rudimentary. However, the question begs to be asked: Are there lessons from the existing ACE qualifications that could inform the process of recurriculation that looms on the horizon?

BACKGROUND AND RATIONALE

Because of the pressing need to have qualified and effective ML teachers in the classroom, the KZN Department of Education (DoE) tasked two HEIs to deliver ACE: ML programmes throughout the province. The in-service teachers, who were prospective students, were identified by the DoE and the HEIs, therefore, had no part in specifying mathematics ability, other than the entrance requirements for the qualification. Each programme spanned two years' of study, and extended to a third year, when students were given an opportunity to repeat modules if they had failed to reach the Universities' required standards. The centres dictated by the DoE were in rural, peri-urban and urban areas.

LITERATURE OVERVIEW

Steen (2003) has written extensively about Quantitative Literacy in the United States. He is of the opinion that learners need to be flexibly prepared for life and to this end he suggests teaching a blend of numeracy, mathematics and statistics. De Lange (2003) posits that being mathematically literate has differing definitions depending on the needs of the community; however, in his description of a balanced mathematical literacy curriculum he identifies topics similar to those in the South African NCS:

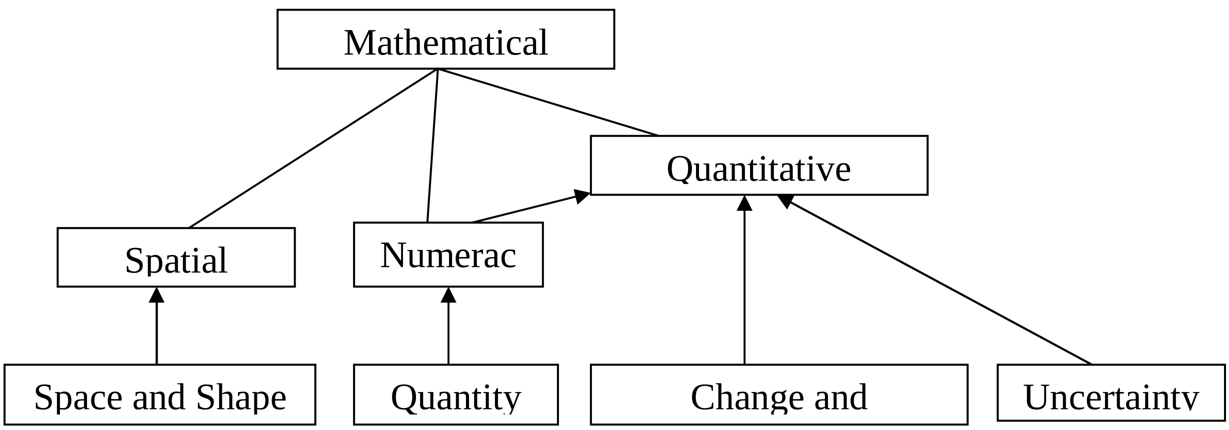

Figure1: Jan de Lange's (2003) conception of a balanced Mathematical Literacy Curriculum

Adler, Pournara, Taylor, Thorne and Moletsane (2009) surveyed the field of mathematics education in South Africa and one of their conclusions was that learning science and mathematics (and, in this instance, mathematical literacy) for teaching is not a simple matter. Many initiatives have not made the required impact on teaching or learner performance. They suggest examining the practices of teacher education with respect to "breadth and depth of domain knowledge; subject content and pedagogy" (p.39)

Schmidt, Cogan and Houang (2011) report on the Teacher Education and Development Study in Mathematics (TEDS-M), an international comparative study of teacher education. The researchers in the TEDS-M study assessed mathematical content knowledge, general pedagogy and mathematical pedagogy as they identified these as three key areas associated with teacher preparation. Although the TEDS-M study focused on the preparation of future teachers, the opportunities to learn are similar to those of in-service teachers who are facing the challenges of a new content area. The TEDS-M study revealed that there is little agreement among HEIs in the United States as to what constitutes teacher preparation, possibly because there is no shared vision of what a highly trained teacher should know (Schmidt et al., 2011). The situation in South Africa with teacher training for ML is similar as there was an urgency to train a massive cohort of teachers before the learning area was implemented in schools, even though the implementation was phased in over three years. HEIs developed programmes in isolation, and perhaps without due regard for Adler et al.'s caveat concerning both breadth and depth of mathematics content. The TEDS-M study further revealed that achievement is related to curricular differences in terms of content cover. Data gathered in the TEDS-M study indicated that the mathematical content offered in teacher training courses was positively related to professional competencies. In the comparison of the two ACEs in this study the majority of modules were ML content-based.

CURRICULUM CONTRASTS

The requirements for entry to ACE: ML for University A recommended that students have at least a pass for mathematics standard grade at Senior Certificate level, a category C (M+3) teaching qualification at NQF level 5 and at least three years' teaching experience. Similarly, University B required an approved initial teacher education qualification or diploma of at least three years' duration and specified that the prospective students should have at least attempted mathematics at Senior Certificate level. In reality 42% of the 1 048 students studying ACE: ML in KZN from University A and 31% of the 691 students from University B had either no mathematics or failed mathematics in their senior certificate examination.

The curriculum for University A consisted of four 30-credit modules. Three modules focused on ML content and were divided into grade levels – grade 10, grade 11 and grade 12 ML content knowledge. The coherence of the curriculum was addressed by using a commercial ML textbook to underpin the curriculum content. The fourth module was designed as an ML pedagogical module that linked content and context.

General pedagogy was deemed to be an aspect of recognition of prior learning (RPL) as the teachers all had a minimum of three years' teaching experience. The emphasis was on reskilling the teachers in a new subject area.

In contrast the curriculum at University B consisted of eight 16-credit modules. The ML content knowledge was covered in four modules that were horizontally aligned to ML curriculum learning outcomes – number and number relationships, functional relationships, space and shape and data handling. The faculty developed their own study material and utilised school textbooks as supplementary material. There was one module devoted to the ML pedagogical aspects of teaching mathematics and mathematical literacy, while another was a research module designed to improve the teachers' reflective practices. In addition, there were two general pedagogy modules, studied by all ACE students in the faculty regardless of the ACE discipline, and two generic research modules –theory and practice of research.

In both universities delivery was through a cascade method of training tutors to teach students throughout the province. The tutors at University A attended a central block of training then went into the field to tutor on Saturdays whereas University B utilised a mixed delivery approach of block sessions and Saturdays. For both ACE programmes modules were delivered over a semester and there was continuous assessment through assignments and tests throughout the semester. In both programmes all modules, except the practical research module at University B, were summatively assessed with an examination. Class marks contributed towards the final examination mark.

PRELIMINARY RESULTS

The statistical data from the study are in the process of being analysed but a few interesting preliminary results are emerging.

The first investigation was to plot the coherence of the curricula according to Steen and de Lange's recommendations. As the South African NCS seems closely aligned to de Lange's diagram both curricula cover all aspects recommended, even though the University B curriculum is horizontally aligned along learning outcomes whereas the University A is vertically aligned according to grades.

The correlation among the pass rate performance of the students in the three ML content modules at University A ranged between 0.751 and 0.826. Correlation was deemed to be significant at the 0.01 level in a two-tailed test. The correlation of the performance in the mathematical pedagogy module as compared with that of the content modules ranged from 0.712 and 0.769. This indicates that if a student achieved well in one of the four modules, there was a strong possibility that the student would also achieve in the other three modules – and vice versa.

At University B the correlation among the performance in the ML content modules ranged from 0.676 to 0.807; however, the correlation was not as strong between the content modules and the ML pedagogical module (minimum correlation was 0.634), the reflective practice research module (minimum correlation was 0.415), and even less between the content modules and the general pedagogy modules (correlation ranged between 0.385 and 0.567). These correlation statistics imply that students who did well in any one of the ML content modules, performed well in the others. However students who did well in the content modules may not have done well in the general pedagogic modules or the research module and vice versa. Thus, these statistics indicate that different skills are required for mathematical content, general pedagogy and reflective practice. Thus Schmidt et al.'s (2011) recommendation is strengthened – that mathematical content knowledge, general pedagogy and mathematical pedagogy should be included in teacher training as they focus on different skills.

Further research is required to ascertain whether the two programmes have made an impact on ML teaching in KZN. The objective of the larger study is to investigate teachers' efficacy in the classroom.

An interesting aspect of the data collection was the link between the students' achievement in mathematics in the Senior Certificate examination and the probability of passing either qualification in the minimum time period of two years. As all the students, registered at both institutions, were in-service teachers with more than three years' experience, they had all written either Mathematics Higher Grade or Mathematics Standard Grade, or had no Mathematics in their final school examination. Their senior certificate symbol for mathematics was converted to a scale variable as depicted in table 2.

Scale variable	Symbol for senior certificate mathematics
0	No mathematics or failed mathematics
1	F symbol for standard grade
2	E symbol for standard grade
3	D symbol for standard grade or F symbol for higher grade
4	C symbol for standard grade or E symbol for higher grade
5	B symbol for standard grade or D symbol for higher grade
6	A symbol for standard grade or C symbol for higher grade
7	B symbol for higher grade

Table 2: Conversion of Senior Certificate mathematics result to scale variable

At both institutions there was a high correlation between the students' mathematics point in the senior certificate examination and their propensity for completing the qualification in the minimum time period. In University A the correlation was 0.805 and in University B the correlation was 0.883. The fact that some students had left school many years before did not appear to affect the data.

CONCLUSION

In this paper we have looked at the composition and student results of two ACE: ML programmes that have been delivered in KZN. The purpose is to learn from the past in order to make informed decisions about programme recurriculation in the near future.

It appears that if students are able to achieve well (or vice versa) in a module focusing on ML content from one learning outcome or grade, this is a reliable indication that they will be able to show good progress (or poor) in modules focusing on other learning outcomes or grades. In this study it appears that the students' ML content knowledge was fairly constant over all learning outcomes and grades in both programmes. Those students who achieved in one content module generally achieved in the other content modules as well.

Another outcome of the study is the positive correlation between mathematics marks in the school leaving examination and the students' achievement in the ACE programmes. Results suggest that despite intervening years, school mathematics marks are a strong indicator of success in completing an ML qualification successfully in the minimum time period. Further interviews are planned with the students which may triangulate this supposition.

The expedience and urgency of delivery precluded including more than curriculum content knowledge in the ML content modules of both curricula. There is ongoing uncertainty and debate about the extent of mathematical depth required to become an effective ML teacher. Should students who studied ML in their senior certificate examination be excluded from training to become ML teachers? If not, what depth of mathematical knowledge should they explore, considering that they do not have a pure mathematics background? However, if ML learners are excluded from ML teacher training, how will South Africa populate schools with ML teachers if the pool of prospective teachers is limited to those who studied pure mathematics successfully at school?

As regards recurriculation, both international and national research has indicated that teacher efficacy is a result of balanced teacher training. The implication is that the areas of ML content knowledge, general pedagogy and ML pedagogy should be included in ML teacher training curricula. The issue of the breadth and depth of domain knowledge has not been addressed; however, it is a pertinent issue to discuss during the recurriculation process. Perhaps the Directorate of Teacher Training could present clear guidelines so that HEIs have an indication of common practice with regards ML teacher training.

The question of vertical academic mobility is a thorny one about which there is little clarity. If the envisaged ACTs do not lead towards further academic qualifications, will the competent teachers be tempted to study to become ML teachers – or will South Africa have to wait for pre-service future teachers to fill the vacant ML posts in schools?

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CREATING NEW MATHEMATICAL STORIES: EXPLORING POTENTIAL OPPORTUNITIES WITHIN MATHS CLUBS

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Math clubs such as the Calculus clubs in the USA have begun pointing to the potential of after school clubs for providing a space for learners (in particular learners from disadvantaged backgrounds)to develop more positive mathematical identities. Math clubs are however an unexplored area of research in South Africa and there is little research (even internationally) that focuses on the potential of maths clubs at early primary level. In this paper I argue for the need to explore this avenue from both a research and development perspective.

INTRODUCTION

Many South African learners and in particular those from poorer backgrounds have negative relationships with mathematics. In this paper I argue that many learner's mathematical histories resonate with experiences similar to those of emotional abuse. Drawing on Sfard & Prusak's (2005) narrative definition of identity – as the reified, significant and endorsable stories we tell, this paper argues that it is critical that we provide the space for learners to develop new mathematical stories and thus identities. Earlier research into identity transformations of mathematical literacy learners paired with my recent experience of starting an FET Maths club in the Eastern Cape point towards the opportunity for the development of more participatory mathematical identities given the opportunity for learner engagement, negotiation and participation (Wenger, 1998).

These experiences lead me to explore, from a research and development perspective, the potential of extra curricula maths clubs, in providing supportive communities where learners can live out different stories. This is not to deny the need for mathematics classrooms to provide the opportunity for more participatory and positive learner identities but rather that the extra curricula nature of such clubs might provide increased freedom to *focus* on the deliberate construction of positive participatory mathematical identities intentionally at the expense of covering the range of skills and knowledge required to 'get through' the curriculum. Thus, for my purpose, I will define maths clubs as extra curricula clubs focused on developing a supportive learning community where learners' active mathematical participation, engagement, enjoyment and sense making are the focus.

My story illuminates that where clubs are formed in such a way that the practice involves deliberate creation of more engaging, confidence building and participatory forms of practice and on the disruption of passive teacher-dependent 'ways of being' learners have an increased opportunity to re-author themselves (and be re-authored) as mathematical producers, mathematical questioners, and mathematical explorers.

Thus at the end of the paper I raise several research opportunities for exploration in relation to what such clubs have to offer that is different or similar to the learning space available in many maths classrooms.

ARE MANY LEARNERS MATHEMATICALLY 'ABUSED'?

In earlier work with Prof Hamsa Venkatakrishnan I explored the nature of shifting learners' mathematical identities in various Mathematical Literacy classrooms from 2006 to 2008 (See Graven, 2009; Venkat and Graven, 2008, Graven and Venkatakrishnan, 2006). Mathematical Literacy was introduced as a new subject in the South African curriculum in 2006 and our research work tracked several learners in the first cohort from Grade 10 (2006) to Grade 12 (2008). In particular that work examined the way in which the new mathematical literacy curriculum enabled the formation of positive mathematical identities and enabled increased access and quality mathematics education particularly for learners with weak mathematical histories. Particularly fascinating in that research was the extent of many learners' negative experiences of mathematics as early as primary school. The experience of mathematics as something completely alien (and alienating) as if they were 'learning in Greek' (as one learner put it) pointed to experiences of an absence of opportunity for sense making and resultant feelings of hopelessness.

Learners' stories regularly included the terms *failure, struggle, stress, nervous, hated maths, worry, extremely difficult, no confidence* and *hopeless*. I include a few excerpts from learner stories of early mathematical experiences written on the last day of their schooling in 2008 in one teacher's (Esme Buytenhuys) mathematical literacy classroom (See Graven and Buytenhuys, 2011). These stories help to illuminate the argument that follows.

> "Ever since Grade 1 I have always struggled in maths. Since I was younger I have refused to do maths homework not because I don't want to do my homework but because I simply did not understand the work that needed to be done."

> "Mathematics for me was a daily struggle. I stressed and cried a lot because of my inability to fully grasp mathematical concepts"

> "Ever since I started school in Grade 1 I have found maths to be one of my real weaknesses. I always struggled and never felt confident. As the years dragged on nothing changed. I felt left behind in every class. It's not fun knowing that there is no hope in the world that you can pass the test."

Many learners also connected their negative mathematical experiences to their broader self-image. For example one learner wrote: "I used to hate anything and everything that had to do with Maths. My struggle with Maths also negatively impacted my self-confidence, and left me feeling like I was stupid and useless".

The wide range of experiences shared by learners in interviews and in their writing led both the teacher and I to consider whether learner experiences of mathematical lessons resonated in any way to experiences of emotional abuse. Esme (the teacher of the class) in writing about the complete mathematical metamorphosis of her learners writes:

> "The caterpillar stage: I clearly remember those first few months with my six Grade 10 Math Lit learners. They started out slinking into my class looking for a place to hide – to go unnoticed for 45 minutes. There was a tangible invisible barrier between the learners and me… I began to understand the nature of the learner who appears in the Math Lit class at the outset of Grade 10. These are precious young people who have been mathematically abused and for most as early as in Primary School." (Graven and Buytenhuys, 2011, 497 – 498).

Searching Wikipedia (2/18/2011) and other definitions of emotional child abuse indicates abuse can include "excessive criticism, inappropriate or excessive demands, withholding communication, and routine labelling or humiliation" and that victims may react by "distancing themselves… internalizing the abusive words… a tendency for victims to blame themselves (self-blame) for the abuse, learned helplessness, and overly passive behaviour". The stories of learners above indicate self-labelling and blame (in terms of mathematical inability), a sense of helplessness, and a distancing and non-participatory stance to further learning. It was interesting to note the relative absence of reference to the teachers' role in the stories although some learners did refer to not understanding their teachers. More commonly the experience of 'excessive criticism' and 'abusive words' came in the form of consistent failing results in work, tests and projects – for example "30% - Fail!". Thus, no matter how unintentional, mathematical learning (or rather lack of learning and failing performance) is experienced by many learners as – 'excessive criticism', 'excessive demands', and 'routine labelling' which is then internalised by learners as 'I am a mathematical failure' or 'I am stupid'.

So while emotional abuse is much broader and more complex than these characteristics of the definition that I have chosen to include here, there is a certain amount of resonance between learner stories and notions of emotional abuse that warrant exploration into the creation of opportunities to work with learners to create new relationships with mathematics and thus to help learners tell new stories about their experiences. Indeed each of the learners above told an entirely different story of their mathematical learning within Esme's Mathematical Literacy class and as a result of active participation and engagement in the class spoke of mathematical competence and confidence to solve mathematical problems in and beyond the classroom. Esme described their metamorphosis into the butterfly stage as becoming mathematical negotiators who no longer shy away from "maths" (See Graven and Buytenhuys, 2011).

WHAT ENABLED LEARNER TRANSFORMATIONS IN ESME'S CLASSROOM?

Overwhelmingly learners' reasons centre around the nature of participation and engagement afforded in these Mathematical Literacy classrooms. Learner comments primarily linked the reasons for this to their changing participation in the classroom in relation to two factors: 'real' collaboration and 'real' problem solving and through these the opportunity for sense making. Furthermore, in this engagement, their own methods and ideas were valued. For Esme (teacher) the slower pace of the curriculum and its explicit reference to getting each learner to 'become a self-managing person' (DoE, 2003, 10) enabled her to focus on developing learner confidence as a priority. She deliberately and consciously worked towards developing learner confidence through encouraging their contributions and following their thinking and methods. In addition the newness of the subject meant that there were no matric assessment precedents to drive the learning in class. Thus there was freedom for Esme to interpret the curriculum's rhetoric of 'maths for life' and 'developing confident mathematical ways of being' according to the needs of the learners in her class without the distraction of preparing for examinations.

Responding to a question in a seminar about how learning trajectories and identity relate to mathematics classrooms where trajectories tend to be outside of the classroom and generally away from Mathematics Wenger[11] posited that perhaps his theory 'was not ready for prime time' for school classrooms. He argued that schools tend to put skills and information before meaning and curricula focus on cognitive aspects of learning while within his theory identity should drive cognition. He writes:

> "What makes information knowledge – what makes it empowering – is the way in which it can be integrated within an identity of participation. When information does not build up to an identity of participation, it remains alien, literal, fragmented, unnegotiable. It is not just that it is disconnected from other pieces of relevant information, but that it fails to translate into a way of being in the world coherent enough to be enacted in practice. Therefore to know in practice is to have a certain identity so that information gains coherence of a form of participation." Wenger (1998, 220)

The learners' early mathematical experiences included above illustrate Wenger's point. On the one hand they reveal that an absence of opportunity to participate meaningfully in mathematical learning alienated learners from developing positive mathematical identities.

Professor Jill Adler at the FirstRand Mathematics Education Chair Community of Practice forum held in Cape town (30/11/2010) argued in her presentation that we have to 'interrupt' the learning and teaching culture in schools where learners are

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[11] On the 8th June 2007 Etienne Wenger visited Wits University, Education Department where he presented a seminar on "Social Learning Theory and communities of practice". This part of the story refers to Wenger's response to a question by Erna Lampen.

passive, learning is teacher dependent and the focus of teaching is on 'compliance' – i.e. passing assessments handed down by the Department of Education (DoE) and producing documentation that can be checked and approved by the DoE. Of course this is a challenge for all teachers and the pressure teachers are under to 'comply' with departmental demands and to meet 'performance standards' is enormous. The rollout of the Annual National Assessments (ANA) in various grades across the country (January/February 2011) and the pressure on schools to perform well in these assessments might of course lead to teaching towards assessment as early as in the Foundation phase. We must indeed find ways within the maths classroom for identity to drive cognition and for increasing learner engagement and participation in mathematical sense making.

However the focus of this paper is on examining the potential of mathematics clubs in supporting the development of positive learner identities outside of the boundaries of curriculum compliance and assessment preparation. Drawing on my own experience of starting a Maths club, I argue that clubs are an opportunity for disrupting passive learning culture and deliberately working with learners to become confident mathematical participators. There is no need in these clubs to 'comply' because they are by definition 'extra-curricular'. This said, I do acknowledge that there might be a tendency for such clubs to comply and to simply be an extension of mathematics classes in after school time. I also acknowledge that there is nothing inherent in an after school maths club (just as there is nothing inherent in mathematical literacy classrooms), that will enable the formation of positive learner identities, only that they hold the potential for providing an enabling space. Perhaps then such clubs can enable the mathematical metamorphosis of learners into mathematical negotiators and producers as we have seen in Esme's class. In order to illuminate the potential of such clubs I share my own story of running a maths club.

MY STORY OF ESTABLISHING AND RUNNING A MATHS CLUB

My story is based on a maths club that I started in February 2009. The club ran from the Mathematics Centre of a private school in South Africa and involved 22 Grade 10 and 11 Mathematics learners (in 2010 they were in Grade 11 and 12) from three 'traditionally disadvantaged' schools that were within 7km of the school. The invitation to learners was through an existing program of Saturday classes that ran with learners in these three schools. The Saturday classes predominantly took the form of whole class teaching that summarised key concepts in relation to various topics in the curriculum. These Saturday classes ended in 2009.

The Maths club ran every Thursday afternoon from 3pm to 4:30pm during term time. In the second year I drew in an additional 'mentor' to support discussion and engagement with learners. Learners tended to arrive at 2pm in order to work together in groups or to use the computers and internet access for research into projects, math games or use of math support programs. Participation in the Maths club was voluntary. Many of the learners had experienced severe disruptions in their mathematical learning at their schools as a result of not having a teacher for substantial periods of time as well as other disruptions such as school specific and nation-wide strikes. The aim of this club was to provide learners a space where they could ask their own questions, produce their own mathematics, talk mathematics, explain mathematics and enjoy mathematics! There would be no ‘on the board’ whole class teaching and learners needed to arrive with their mathematical productions and ideas about what they wanted to work on. I positioned myself as a mentor rather than the teacher - this was made easier by the fact that I was not an FET mathematics teacher. They had to show and explain to me what they had produced and engage me on where they felt they were confused. They needed to take the lead in asking me questions related to their mathematical activity over the past week. No activity plan, lesson plan or homework plan was given for the club. Each session was entirely based on what learners brought to the session and what they chose to engage with. A range of resources were provided including some text books on newer curriculum topics and a set of past examinations with solutions. New calculators were also provided thanks to a donation by Casio. Learners were provided with the “Ask Archie” computer learning program that ran through key concepts in each of the topics covered in Grade 11 and Grade 12. Learners were able to use these at their schools, in community centres or in the Mathematics Centre before and after club sessions.

The club also became a space for negotiating future possible mathematical trajectories through discussing and investigating career options. Learners were provided access to resources such as the UNISA College of Science, Engineering and Technology MathEdge CD and website *www.MathsEdge.org.za* which provides access to critical information on mathematical careers, local university mathematical courses, access to entrance requirements and stories of inspirational achievements of South Africans working in mathematical fields. One of the learners, Vusi (pseudonym), drew on the club to explore post school mathematics that he had come across. He had heard about a topic taught at University and wanted to know how to do it. We researched some of it over the internet (as my memory of some University maths was shaky) and he went off with my first year varsity mathematics textbooks. He dived into Olympiad type questions and tended to use the time in the club to find more challenging work. He showed little interest in the core maths curriculum taught at his school. He also used the club as a space where he could show his alternative methods to solving a problem and seemingly enjoyed convincing us ‘mentors’ of the logic of his methods as well as the beauty and sometimes superiority of his approach. His focus was to use the club to extend and challenge himself and to share findings and methods that were not in the textbook. His novel ways of approaching problems were interesting and created learning opportunities for me as I began to understand some mathematical topics in new ways. While he showed little interest in exam preparation he nevertheless achieved 100% for his 2010 matric national mathematics examination and has gone on to study a B Sc with a specialisation in Mathematics with a full bursary. While this student is of course somewhat exceptional the other three participating Grade 12s achieved two B's and a D in their matric examination. The remaining students in the club will complete their Grade 12 this year in 2011.

The point for me about this club was that rather than teaching 'catch up' lessons in whole class style to learners (as in the previous Saturday classes) the club enabled learners to take responsibility for their own learning and for mentors to support their mathematical journeys. For Vusi it was about extending himself way beyond what was in the curriculum and the opportunity for being challenged and sharing/communicating his discoveries of how to solve something in a new way and having to defend and argue these methods. For others it was about engaging support for understanding work they had been given at school and had tried themselves, often with other members of the club. As with Vusi they would lead the learning with their questions that emerged from their individual or small group mathematical workings and productions.

While I have no doubt that Vusi would have succeeded at Mathematics irrespective of this club, (Vusi obtained the top result, a B, for an assessment conducted by the teacher of the Saturday classes program in early 2009), the club enabled a strengthened identity as an excellent mathematical thinker and someone who produced his own mathematics. It supported him in achieving exceptional results which resulted in his identification as a top mathematical achiever both locally and regionally thus strengthening the development of a future mathematical trajectory. (He is majoring in Mathematics in his B Sc with his eye on possibly going into actuarial science). Similarly the results of the other participating students showed improvement. The improving performance, however, is less an indicator of the success of the club than the more active, sense making and participatory mathematical ways of working that learners adopted.

WHAT FRAMES ARE USEFUL IN REFLECTING ON MY STORY?

I have found Wenger's (1998) seminal work on communities of practice particularly useful for thinking about the learning community of a maths club and the importance of identities of participation (and non-participation). However to research learner identities unfolding through participation in a community of practice such as math clubs I need an operational definition of identity. In order to provide an operational definition for identity Sfard and Prusak (2005, 16) define identities as "collections of stories about persons or, more specifically, as those narratives about individuals that are reifying, endorsable, and significant". Reification comes with verbs such as 'have' (e.g. 'I have a mathematical brain') and I would add with declarations of one's being such as 'I am' (e.g. 'I am mathematically stupid'). Stories are considered endorsable if the identity builder can answer to them being a faithful reflection of a state of affairs. (e.g. 'I'm mathematically stupid because I always fail tests'). Stories are *significant* if a change in the story is likely to affect the storyteller's feelings about the identified person - e.g. a change in the story that ‘he is a slow learner’ to ‘he thinks deeply about each problem’ is likely to lead to a change in feeling by the storyteller about learners.

Thus, within their definition identities are human made, collectively shaped by authors and recipients. This definition is helpful to researchers as this operational definition means we can access these identities through interacting with learners and teachers and paying attention to the stories told. Their definition differs from Wenger’s (1998) notion of identity in the sense that Wenger sees these discursive counterparts as only part of "the full, lived experience of engagement in practice". Despite this divergence Wenger’s perspective on identity is particularly useful in considering the *process* of the formation of identity through his three modes of belonging within a community of practice: engagement, alignment and imagination. These modes of belonging in conjunction with identification and negotiability (as the mechanisms by which modes of belonging become part of our identities) enable us to explore how learner identities are shaped within the broader framework of learning within a community of practice.

Freedman and Combs (1996) argue that the metaphor of stories helps one to see how stories circulate in society and how these realities are socially constructed, constituted through language and organized and maintained through narrative:

> "When life narratives carry hurtful meanings or seem to offer only unpleasant choices, they can be changed by highlighting different previously un-storied events, thereby constructing new narratives. Or when dominant cultures carry stories that are oppressive, people can resist their dictates and find support in subcultures that are living different stories" (p32-33).

In this sense supportive communities such as maths clubs hold the potential for enabling learners to live different stories. These maths clubs should open up these alternatives because as we have seen in many of the learner stories included in this paper these stories ‘carry hurtful meanings’, undermine mathematical identities and impede learning. The stories of learners in Esme’s class and in the math club indicate a willingness and capability of learners to re-author their mathematical identities given the opportunity for mathematical sense making and active participation, engagement and negotiability. It is hoped that the future research and development work into learner clubs will provide a fruitful space for engagement leading to shifts in the deficit discourse of numeracy education by highlighting opportunities available within the current landscape of educational constraints.

WHERE TO FROM HERE?

What has been described above is my story about stories (Sfard and Prusak, 2005). I have told my story of my experience in this club with the view to informing my own trajectory of working with clubs in primary schools within the FirstRand Foundation Chair in Numeracy Education. I believe that this story indicates the potential for what mathematical clubs might offer given the creation of learner opportunities for active engagement, negotiation and participation. Furthermore, since negative non-participatory learner identities seem to appear in many learner stories quite early in their mathematical learning it would make sense to explore the opportunity clubs might offer in 'interrupting' these negative relationships and re-authoring new positive experiences, stories and thus identities. The danger of negative stories (negative labeling) is they become self-fulfilling prophecies (Sfard and Prusak, 2005) and hence shut down the space for future learning and create cul de sacs along potentially exciting mathematical learning trajectories.

In beginning my learning trajectory into maths clubs I define, for our purposes here, maths clubs as extra curricula clubs focused on developing a supportive learning community where learners active mathematical participation, engagement and sense making are the focus. Individual, pair and small group interactions with mentors are the dominant practices with few whole class interactions. From this point of departure the following questions emerge as potentially useful in mapping a future research trajectory:

Can Maths clubs interrupt passive teacher dependent learning identities of primary learners? If so what are the mechanisms that allow for this?

How might opportunities for learner independence, argumentation and listening to others perspectives (negotiability) differ for differing educational bands? differ for learners with differing mathematical confidence[12] levels?

Can maths clubs provide the space for mathematical support directed at help for learners to overcome their fear, shame and helplessness? If so how can clubs enable this? What norms, practices, activities, mentor and peer relations and forms of mediation are needed in clubs to allow space for such support?

Who might be drawn on as mentors for such clubs? How might mentors be prepared for their role in such clubs?

What research lenses, analytical tools and related methodological approaches might be useful in researching these questions?

A general scan of the literature in the field of mathematics education in Southern Africa indicates an absence of literature pertaining to learning within math clubs and furthermore international literature relating to clubs tends to focus on high school learners. We hope that we can contribute towards initiating engagement on the potential of such clubs both nationally and beyond in order to enter into dialogue in relation to the above questions and others that arise through this investigation.

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[12] Within the perspective taken in this paper I replace the much used term 'ability groups' with the term confidence groups as especially in the early years and with labels resulting in self-fulfilling prophecies I believe it is more useful to think about and characterize learners in terms of their confidence levels demonstrated through their levels of engagement and participation than referring to some reified ability that is considered to reside inside of learners.

Acknowledgements

The proposed research and development work is supported by the South African Numeracy Chair Initiative of the FirstRand Foundation (with the RMB), Anglo American Chairman's fund, Department of Science and Technology and the National Research Foundation

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REFLECTING ON THE VAN HIELE THEORY

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This paper gives a brief overview of the Van Hiele Theory of learning geometry, and highlights and illustrates some important theoretical implications for designing learning activities, not only in dynamic geometry contexts, but also traditional textbook driven contexts. Three problematic issues for possible further research are suggested such as hierarchical class inclusion, the greater involvement of learners in defining the quadrilaterals themselves, and the development of understanding of functions of proof other than just verification.

INTRODUCTION

The Van Hiele theory originated in the respective doctoral dissertations of Dieke van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. A distinctive feature of the theory is that children's understanding of geometry progresses through distinct levels, and mastery of each level is a prerequisite for progression to the next.

The differences between the first three levels can be summarized as shown in Table 1 in terms of the objects and structure of thought at each level (adapted from Fuys, Geddes & Tischler, 1988:6).

	Level 1	**Level 2**	**Level 3**
Objects of thought	Individual figures	Classes of figures	Definitions of classes of figures
Structure of thought	Visual recognition Naming Visual sorting	Recognizing properties as characteristics of classes	Noticing & formulating logical relationships between properties
Examples	• Parallelograms all go together because they "*look the same*" • Rectangles, squares and rhombi are not parallelograms because they do "*not look like them*"	A parallelogram has: • 4 sides • opposite angles = • opposite sides = • opposite sides // • bisecting diagonals; etc.	• Opposite sides = imply opposite sides // • Opposite sides // imply opposite sides = • opposite angles = imply opposite sides

	A rectangle is not a parallelogram since a rectangle has 90° angles, but a parallelogram not.	= • bisecting diagonals imply half-turn symmetry

Table 1: Objects and Structures of Thought at different Van Hiele levels

THE PRIMARY & MIDDLE SCHOOL GEOMETRY CURRICULUM

In South Africa we used to have a geometry curriculum heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school. (E.g. little similarity or circle geometry is done in the primary school). On average, learner' performance in the South African matric (Grade 12) geometry has been far worse than in algebra as evidenced by the yearly reports of the Chief Examiner.

The Van Hiele theory supplies an important explanation for this discrepancy. For example, research by De Villiers & Njisane (1987) showed that about 45% of pupils investigated in Grade 12 in KwaZulu had only mastered Level 2 or lower, whereas the examination assumed mastery at Level 3 and beyond! In particular, the transition from Level 1 to Level 2 posed specific problems to second language learners, since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This requires sufficient time, which is not available in the presently overloaded secondary curriculum.

In Japan, for example, pupils already start off in Grade 1 with extended tangram, as well as other planar and spatial, investigations (e.g. see Nohda, 1992). This is followed up continuously in following years so that by Grade 5 they are already dealing formally with the concepts of congruence and similarity; concepts that are only introduced in Grades 8 and 9 in South Africa. Similarly in Taiwan where geometry is started early, it is reported in a study by Wu & Ma (2006) that 28.3% of Grade 6 learners were already at Van Hiele 3, whereas the same percentage of learners at Van Hiele Level 3 in South Africa, only occurred in Grade 11 (De Villiers & Njisane, 1987). More recently, Feza & Webb (2005) found that only 5 out of 30 (16.7%) Grade 7 learners interviewed in South Africa, had reached Van Hiele level 2. It seems no wonder that in international comparative studies in recent years, Japanese and Taiwanese school children have consistently outperformed school children from South Africa, as well as other countries.

Although the recent introduction of tessellations in South African primary schools is to be greatly welcomed, many teachers and textbook authors do not appear to understand its relevance in relation to the Van Hiele theory. Although tessellations have great aesthetic attraction due to their intriguing and artistically pleasing patterns, the fundamental reason for introducing it in the primary school is that it provides an intuitive visual foundation (Van Hiele 1) for a variety of geometric content, which can later be treated more formally in a deductive context.

For example, in a triangular tessellation pattern such as shown in Figure 1, one could ask pupils the following questions:

(1) identify and colour in parallel lines

(2) what can you say about angles *A*, *B*, *C* , *D* and *E* and why?

(3) what can you say about angles *A*, *1*, *2*, *3* and *4* and why?

In such an activity pupils could realize that angles *A*, *B*, *C*, *D* and *E* are all equal since a halfturn of the grey triangle around the midpoint of the side *AB* maps angle *A* onto angle *B*, etc. In this way, pupils can be introduced for the first time to the concept of "*saws*" or "*zig-zags*" (alternate angles). Similarly, pupils should realize that angles *A*, *1*, *2*, *3* and *4* are all equal since a translation of the grey triangle in the direction of angles *1*, *2*, *3* and *4* consecutively maps angle *A* onto each of these angles. In this way, pupils can be introduced for the first time to the concept of "*ladders*" (corresponding angles). Learners should further be encouraged to find different saws and ladders in the same and other tessellation patterns to improve their visualization ability.

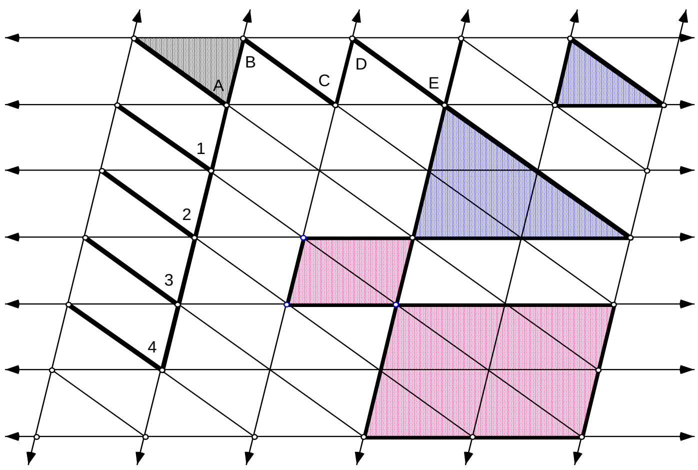

Figure 1: Visualization

Since each tile has to be identical and can be made to fit onto each other exactly by means of translations, rotations or reflections pupils can easily be introduced to the concept of congruency. Pupils can also be asked to look for different shapes in such tessellation patterns, e.g. parallelograms, trapezia and hexagons. They could also be encouraged to look for larger figures with the *same shape*, thus intuitively introducing them to the concept of *similarity* (as shown in Figure 1 by the shaded similar triangles and parallelograms).

Conceptual Structuring

A very important aspect of the Van Hiele theory is that it emphasizes that informal activities at Levels 1 and 2 should provide appropriate "*conceptual substructures*" for the formal activities at the next level.

Teachers often let their students measure the angles of a triangle with a protractor, and then let them add the angles (usually disregarding ‘deviations’ as due to experimental error) to ‘discover’ that they always add up to 180°.

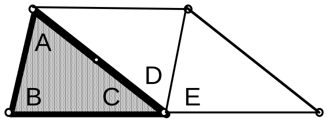

Figure 2: Using transformations to discover

However, from a Van Hiele perspective this is entirely inappropriate as it does not provide a suitable conceptual substructure in which the eventual logical explanation (proof) is implicitly embedded. In comparison, an activity with cardboard tiles or *Sketchpad* like the following from De Villiers (2003) provides such a substructure. For example, translate a triangle *ABC* by vector *BC*, and rotate triangle *ABC* around the midpoint of *AC* (see Figure 2). Let the students notice through dragging that the three angles at *C*, *D*, and *E* always form a straight line. Then ask students what they can say about angles *A* and *B* in relation to angles *D* and *E* in terms of the transformations carried out. Since angle *B* maps on to angle *E* by the translation, and angle *A* maps to angle *D* by the half-turn, angles *B* and *A* are equal to angles *D* and *E*, respectively. Clearly this provides a much more appropriate conceptual structure for an eventual explanation (proof) than simply letting students measure some angles of triangles.

Defining and classifying

Traditionally most teachers and textbook authors have simply provided students with ready-made content (definitions, theorems, proofs, classifications, and so on). However, just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is that not all angles or sides are allowed to be equal.

According to the Van Hiele theory, understanding of formal textbook definitions only develops at Level 3, and that the direct provision of such definitions to students at lower levels would be doomed to failure. In addition, if we take the constructivist theory of learning seriously (namely that knowledge simply cannot be transferred directly from one person to another, and that meaningful knowledge needs to be actively (re)-constructed by the learner), students ought to be engaged to some extent in the activity of defining and allowed to choose their own definitions at each level (compare De Villiers, Govender & Patterson, 2009). Dynamic geometry software makes it easier to experimentally check the validity of definitions by construction and dragging, and results from Govender & De Villiers (2003) and Sáenz-Ludlow & Athanasopoulou (2007) do indicate some improvement and positive gains in student understanding of the nature of definitions when they are involved in the process, as well as in their ability to define quadrilaterals themselves.

Hierarchical versus Partition Definitions

Though children at an early age are capable of understanding class inclusions like "*cats and dogs are animals*", it appears substantially more difficult to accomplish with geometric figures. Generally, students' spontaneous definitions at Van Hiele Levels 1 and 2 would also tend to be *partitional* as shown above in Table 1; in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van Hiele theory, definitions at Level 3 are typically *hierarchical*, which means they allow for the inclusion of the squares among the rectangles, and would not be understood by students at lower levels.

In traditional instruction children are mostly introduced to rectangles, rhombi, parallelograms, etc. as '*static geometric objects*'. For example, a rectangle might be introduced by comparison to the shape of a door or a static picture in a book, but a door or a picture in a book cannot be transformed into a square (unless parts are cut off). So the concept rectangle is from the start introduced as a concept completely disjoint from a square. Unfortunately this partition classification schema then becomes entrenched and fossilized over time, and appears very resistant to change.

The conceptual difficulty of geometric class inclusion was already shown by Mayberry (1981) who found that only 3 out of 19 preservice mathematics teachers indicated squares also as rectangles on a sheet of some given quadrilaterals. In research conducted by De Villiers & Njisane (1987) with 4015 students from KwaZulu (South Africa) it was found that very little progress occurred in their hierarchical thinking from Grade 9 to Grade 12, only ranging from 0.5% to 5.1% success with a 50% criterion on test items evaluating hierarchical thinking. This contrasts starkly with Van Hiele 3 proficiency levels in one-step and two-step deductions that respectively improved from 2.5% and 0.2% in Grade 9 to 63.3% and 42.6% in Grade 12. More recent findings by Atebe & Schäfer (2008) with a group of Nigerian and South African similarly showed that class inclusions of quadrilaterals among the investigated group from Grades 10-12 were almost completely absent.

One common difficulty students have in producing correct counterexamples to incomplete definitions is that they often try to refute a definition with a special case. For example, for the incorrect definition "*a rectangle is any quadrilateral with congruent diagonals*," some students will provide a square as a counterexample. But obviously a square is not a valid counterexample, because a square *is* a rectangle.

Therefore, students should already have developed a sound understanding of a hierarchical (inclusive) classification of quadrilaterals before being engaged in formally defining the quadrilaterals themselves (Craine & Rubenstein, 1993; Casa & Gavin, 2009). This development can be fostered by using interactive geometry software, figures created with flexible wire, or paper-strip models of quadrilaterals.

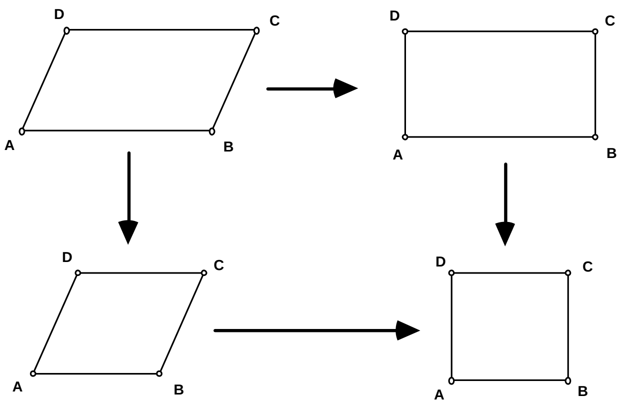

Figure 3: Dynamic transformation of parallelogram

Specifically, the dynamic nature of geometric figures constructed in dynamic software like *Sketchpad, GeoGebra, Autograph, Cinderella* or *GeoNext* may make the acceptance of a hierarchical classification of the quadrilaterals far easier at lower Van Hiele levels. For example, if students construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 3. In fact, it seems quite possible that at least some students would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed. It is quite possible too that student difficulties with hierarchical class inclusion is largely the result of traditional instructional practices, something already observed by Mayberry (1981:8) when she wrote: "*It is conceivable that the observed levels are an artifact of the current curriculum or the instruction given to the students …*"

Construction and Measurement

It should first be pointed out that certain kinds of construction activities (with dynamic geometry software or by pencil and paper) are inappropriate at Van Hiele Level 1. For example, at Van Hiele Level 1 it is far more appropriate to provide children with ready-made sketches of quadrilaterals in dynamic geometry software, which they can then easily manipulate and first investigate visually than to expect to construct them by themselves (also compare Battista, 2007). Next, they could start using the measure features of the software to analyze the properties (and learn the appropriate terminology) to enable them to reach Level 2. Only then would it be appropriate to start challenging learners to construct such dynamic quadrilaterals themselves, thus assisting the transition to Level 3.

However, Level 2 students cannot yet be expected to logically check their own descriptions (definitions) of quadrilaterals, but they should be allowed to do so by accurate construction and measurement. Conceptually, constructions are extremely important for assisting the transition from Van Hiele Level 2 to Van Hiele Level 3. It helps to develop an understanding of the difference between a *premise* and *conclusion* and their *causal* relationship; in other words, of the logical structure of an "*if-then*" statement (compare Smith, 1940).

CONCLUDING COMMENTS AND QUESTIONS

Traditionally, the development of 'proof ability' is seen to occur from Van Hiele 3 Level onwards. However, the Van Hiele model sees proof mainly as a means of 'verification', and it remains an open research question whether or not other functions of proof such as 'explanation' can be utilized and developed earlier at the visual and analytic Levels 1 and 2 respectively (see for example Mudaly & De Villiers, 2000). Can more explanatory, visual-dissection proofs and arguments by symmetry (line, rotational, point) be developed and understood by children at lower Van Hiele levels?

It would appear that the 'systematization' function of proof as distinguished by Bell (1976) would fit locally on Level 3, and more globally on levels 4 and 5, but it's not quite clear where understanding of the 'discovery' function of proof as distinguished by De Villiers (1990) would fit into the Van Hiele model.

Lastly, it seems one of the major outstanding research problems on the Van Hiele theory is the issue of hierarchical thinking (class inclusions). Some questions remain: Why is partition thinking so persistent once a person has such a view? Is partition thinking merely the consequence of traditional geometry teaching strategies, or is it unavoidable, irrespective of the teaching approach used? The author, for example, believes that hierarchical thinking, or at least the seed for it, could potentially be planted and developed before Van Hiele level 3 at Van Hiele levels 1 and 2 possibly as early as Grades 1 and 2, especially through using tools such as dynamic geometry software by dragging strategies as suggested in Figure 3. For example, using moveable wire models to demonstrate class inclusion of quadrilaterals to young Grade 5 learners, a study by Malan (1986) has already shown promising results, and are supported by more recent ones by Erez & Yerushalmy (2006) and Gal & Lew (2008) with young learners.

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PEDAGOGICAL DESIGN CAPACITY (PDC): PANACEA FOR UNDERSTANDING TEACHER-TEXT RELATIONSHIP

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This article reports on a survey of research on the construct of Pedagogical Design Capacity (PDC). The survey is based on the works of two authors, Matthew Brown (Brown, 2002, 2009), and Janine Remillard (Remillard, 2005, 2009). While Remillard offers a framework for studying teacher-text relationships, Brown's PDC fills the gaps in the aforementioned framework that enhances our understanding of the teacher-text interactions. The article argues that PDC is an appropriate route to understanding how teachers engage with their textbooks, illustrates why orientation to teachers' use of textbooks is important, and demonstrates why harnessing the pedagogical design capacity route is potentially productive.

> The song Take the A Train, written by Billy Strayhorn, was the signature tune of the Duke Ellington Orchestra, and was performed by countless others. If we compare Duke's rendition to one by Ella Fitzgerald, we have little difficulty identifying each rendition as being the same song. Yet, despite their essential similarities, the songs sound distinctly different. …This relationship is similar with curriculum materials and teacher practices.
>
> In both cases, practitioners bring to life the composer's initial concept through a process of interpretation and adaptation, with results that may vary significantly while bearing certain core similarities. Just as modern music has come to rely on sheet music as a representational medium for conveying musical concepts, forms, and practices…, classroom instruction has come to rely on curriculum materials as tools to convey and reproduce curricular concepts, forms, and practices….
>
> In both cases, no two renditions of practice are exactly alike.
>
> Matthew W. Brown (2009)

INTRODUCTION

My interest in textbooks and how teachers use them was sparked by observing two teachers using textbooks in their classrooms. Both teachers used the textbooks substantially but in quite different ways. The first teacher was using the textbook for exercises, interestingly skipping some, and doing others in full, while the second teacher used the textbook mainly for formulas and definitions, and also bringing in other books. What I also observed was that the teachers were using the textbooks in quite ad-hoc ways. I have to say here that I only observed one lesson with each teacher, and all that is being said here is on the basis of a one-off observation, which may not be the case in the two teachers' daily practice. But the two observations set me thinking about how teachers use textbooks and why they use them in the way that they do.

The opening quotation by Matthew Brown illustrates some facts which we already know; that mathematics teaching relies a lot on textbooks and curricular materials as the major tools of the trade; that different teachers use textbooks differently; but most importantly impels me to want to understand why teachers do what they do with their textbooks.

The article starts with a historical background of research on mathematics textbook use, placing emphasis on the context of the article. It then proceeds into Remillard’s (Remillard, 2009) proposed theoretical framework for studying the teacher-text relationships, elaborating on the characteristic features of these interactions. Then follows the notion of Pedagogical Design Capacity and a discussion of why PDC adds value to illumination of the teacher-text interaction.

RESEARCH IN MATHEMATICS TEXTBOOKS USE

Mathematics textbooks remain a major resource for teaching from developing to the most developed of countries, and are an intricate part of what is involved in doing school mathematics (Askew, Hodgen, Hossain, & Bretscher, 2010; Haggarty & Pepin, 2002; Johansson, 2006; Moulton, 1997; Nicol & Crespo, 2006; Remillard, 1999; Vincent & Stacey, 2008). However researchers in the field are of an opinion that there is dearth of research on textbooks use worldwide (Askew, et al., 2010; Moulton, 1997). Hodgen, Kuchemann & Brown (2010), note that even though “textbooks are almost ubiquitous in mathematics classrooms across the developed world and are amongst the most influential factors in the implemented curriculum” (p. 2), research on mathematics textbooks is much more limited than research investigating the implementation of curriculum resources.

It is worth noting that the field of research studies investigating the use of textbooks and other curricular resources for teaching and learning mathematics has grown in the past two decades, and is gaining momentum with the publishing of the first compilation of its kind in 2009 (Lloyd, Remillard, & Herbel-Eisenmann, 2009) and possibly another coming in 2011 on the subject of teachers engaging with curricular resources. However, much of the growth hails from the Unites States (US) and focuses especially on the use of curriculum materials. These materials are based on the *Standards* ("National Council of Teachers of Mathematics [NCTM]," 1989)and funded by the National Science Foundation (NSF). They are reform-oriented and designed to support teacher change, hence may not be similar to the traditional textbook as we know it. As Lloyd et al.(2009) point out; the materials “contain mathematical emphases (eg mathematical thinking and reasoning, conceptual understanding, and problem-solving in realistic contexts) and pedagogical approaches that were previously uncommon in textbooks published in the United States” (p. 4). However, in this article we shall use ‘curriculum materials’, ‘textbooks’, and ‘curricular resources’ to refer to the same thing.

Context

This article constitutes a survey of research on teachers’ relationships with texts, and what might be productive ways to understand these. I will focus in particular on the works of two researchers: Remillard (2005, 2009), who in response to questions about teachers' relationships with texts compiled syntheses of research findings on teachers' use of mathematical curricular resources and drew up theoretical frameworks for studying and understanding the field better; and Brown (2005,2009), who constructed the notion of Pedagogical Design Capacity (PDC) as a concept that might be useful for thinking about what it is that enables us to understand why teachers use curricular resources differently.

Remillard suggests that three constructs of curriculum use, teaching, and curriculum materials are central to this body of research, and that the way these constructs are conceptualised have impact on knowledge in the field. A synthesis of research yielded four ways 'use' was conceptualised and examined in research (Remillard, 2005):

a) use as *following or subverting* the text - the teacher is viewed as a "conduit", as just the "faithful implementer of the "ideal" curriculum that represents a "pure" interpretation of author's intended designs for teaching and learning sequences as seen in the materials they are asked to implement"(Ziebarth, et al., 2009, p. 172). The authority for both the mathematics that is to be taught and the sequencing and presentation of content are given to the text, and strict adherence to the text becomes the goal of teaching (McClain, Zhao, Visnovska, & Bowen, 2009).

b) Use as *drawing on* the text - The emphasis here is on the agency of the teacher and views texts as just one of the many resources that teachers use in constructing the enacted curriculum. Fidelity is accepted as a possibility by some researchers while others do not accept it at all.

c) Use as *interpretation* of text - In this view the teacher is framed as an interpreter of the written curriculum. Fidelity between classroom action and written words in a teacher's guide is impossible. This means that teachers bring their own beliefs and experiences to their encounters with curriculum to create their own meanings, and that by using curriculum materials teachers interpret the intentions of the authors. Chavez (2003) asserts that "it is possible to adopt a textbook and use it frequently without really espousing the epistemological assumptions that are attached to the textbook, and thus not change teachers' practices in ways that would better match the goals of a particular curriculum" (p.160)

d) Use as participation with text - This view seems a "less common perspective taken by researchers studying teachers and curriculum materials" (Remillard, 2005, p. 221). It focuses on the teacher-text relationship, or the activity of using the text, and treats use of curricular resources as collaboration with the materials. The view attends to particular features of the text as well as teachers' interpretation of those features (Stein & Kim, 2009), and is centred on the assumption that "teachers and curriculum materials are engaged in a dynamic interrelationship that involves participation on the parts of both the teacher and the text" (Remillard, 2005, p. 221). In this dynamic interrelationship, each

> participant (teacher and text) shapes the other, and together they shape the teaching. Remillard goes further to point out that "the distinguishing characteristic of this perspective is its focus on the activity of using or participating with the curriculum resource and on the dynamic relationship between the teacher and curriculum". As such, studies in this view "not only look at how teachers engage with, use, shape, adapt, and interpret curriculum materials but also consider how teachers change or learn from their use of these resources"(Remillard, 2005, p. 222).

This article aligns itself with the view of textbook use as participation with text in seeking to understand this dynamic interrelationship between teacher and text.

Brown on the other hand, claims that in order to understand the dynamic interplay that unfolds when teachers use curriculum materials"(p. 23)**,** teaching needs to be framed as a "*design process"*(Brown, 2002, p. 1). Teaching as design captures "the creative and in-process characteristics of teaching" (Remillard, 2005, p. 224) which views teachers' work as much more than "curriculum development" (Ben-Peretz, 1990)

> teaching involves a particular brand of design. When teachers use curriculum materials to craft instructional episodes in order to achieve goals, when they use materials as tools to transform a classroom episode from an existing state to a desired one, they are engaging in design – whether or not they intend to do so. Whether teachers modify an existing set of materials or integrate them in a literal manner, they are engaging in the sort of goal-directed activity I am calling design (Brown, 2009, p. 23)

Brown further posits that a primary challenge that teachers face in using curriculum materials is "developing the capacity to perceive, interpret, and mobilise the features of the designs in the pursuit of desired outcomes" (Brown, 2002, p. 24), the development of teachers' Pedagogical Design Capacity.

THEORETICAL FRAMEWORK FOR UNDERSTANDING THE TEACHER-TEXT RELATIONSHIP

The theoretical background for Brown's (2002, 2009) research work on teachers' use of curriculum materials is situated within sociocultural theory wherein, all humans are inherently social beings and grow from and through the use of tools (Vygotsky, 1978). The emphasis is in understanding the processes of mediated action as derived from Vygotsky and developed by Wertsch (Wertsch, 1991, 1998). Wertsch (1991) shows that the cultural tools that are employed in mediated action are the key to understanding the relationship between sociocultural settings and human action. The main function of a cultural tool according to Vygotsky (1978)

> is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to changes in objects. It is a means by which human external activity is aimed at mastering, and triumphing over, nature (p.55)

These tools or artefacts, which are "products of sociocultural evolution" (Wertsch,

1998) are created by humans to assist them to achieve goals which they could not accomplish on their own. They also mediate activity through their affordances and constraints (Norman, 1988; Pea, 1985; Vygotsky, 1978; Wertsch, 1991, 1998).

Remillard (2009) offers a framework for understanding the teacher-text interactions and relationships within this theoretical setting.

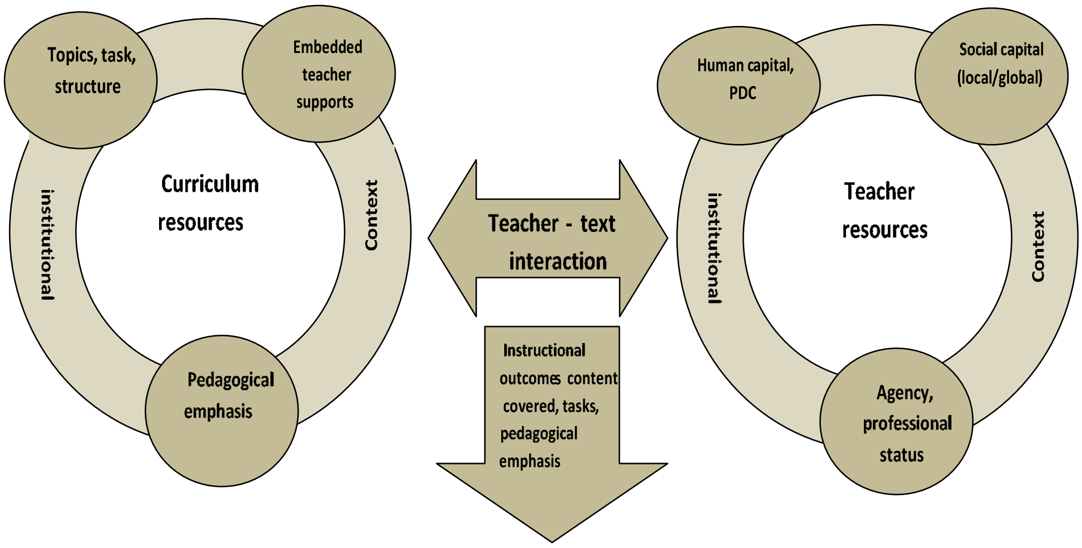

Figure 1: Conceptual Model of Teacher – Curriculum Interactions and Relationships

The model proposed is a synthesis of research work in the field and also encompasses the Design Capacity Enactment Framework (Brown, 2002) and some of the elements from the Framework of Component of Teacher-Curriculum Relationship (Remillard, 2005). The model identifies and classifies components of teacher resources and curriculum resources which enter into the dynamic interrelationship between teacher and text to produce instructional or teaching outcomes. The textured rings surrounding the curriculum resources and teacher resources represent features of the institutional contexts which have been found to greatly influence how teachers interact with curricular resources (McClain, et al., 2009; Stein & Kim, 2009).

Teacher Resources

Teacher resources in the interation include:

a) agency and professional status of the teacher, to refer to how teachers locate themselves within a school system and the authority with respect to decision-making (Stein & Kim, 2009);

b) social capital, comprising local teacher networks and global social systems (Stein & Kim, 2009);

c) human capital which includes teacher knowledge such as subject matter and pedagogical content knowledge, and beliefs. It also includes *pedagogical design capacity* (PDC), the main object of this article which will be discussed in the next secton.

Curriculum/Textbook Resources

Curriculum resources comprise:

a) topics, task,and structure, to describe the inclusion of mathematical topics and tasks that are stuctured in deliberate ways (Brown, 2009; Chval, Chavez, Reys, & Tarr, 2009; Stein & Kim, 2009)

b) embedded teacher supports, which are intended to guide pedagogical decision-making (Stein & Kim, 2009)

c) pedagogical emphasis of the materials, found in teaching strategies and lesson structures (Chval, et al., 2009)

The proposed framework illuminates two critical issues about the teacher-text interactions. Firstly, these are at the centre of the model and influence what kinds of instructional outcomes the interactions are producing. Secondly, it shows that the teacher-text relationship is a two-way dynamic process. Neither the textbook, nor the teacher should be regarded as the final authority on what finally gets enacted in the classroom. This is a relationship in which two participants have to communicate with each other and shape each other in order for the teacher to end up with a product that is responsive to the students' needs and the teaching goals.

However, Brown (2002) makes an assertion that, understanding which features of curricular and teacher resources come together in shaping the outcomes of the teacher-text interactions does not sufficiently explain similarities and differences in teachers' use of materials. This insufficiency of the model to explain, and describe the process of interaction is also echoed by Remillard (2009): "the chapters…offer insights into the factors that influence the teacher-text relationship, but provide few details about its nature"(p.89)

To address this shortfall and to advance research in understanding the skills and processes by which teachers identify and mobilise such resources when crafting classroom activities, and to develop a way of characterising such processes, Brown introduces the notion of *pedagogical design capacity* (PDC).

THE NOTION OF PEDAGOGICAL DESIGN CAPACITY

As mentioned in the previous section, knowing which features of curricular resources and teacher resources play important roles in shaping the outcomes of the teacher-text interactions is not sufficient in explaining similarities and differences in teachers' use of materials. The challenge in the field is to identify the ways in which teachers interpret and mobilise these resources in the classroom, that is, to identify not just the types of resources teachers use but also the skills and processes by which teachers identify and mobilise such resources when crafting classroom activities, and develop a way of characterising such processes.

In order to address this challenge, Brown develops the notion of *Pedagogical Design Capacity* (PDC) that characterises teachers' capacities to use specific personal and material resources to craft classroom episodes. *Pedagogical design capacity* is defined as *"teachers' capacity to perceive and mobilise existing resources in order to craft instructional contexts" (p.70).* PDC is a theoretical construct emanating from "a vision of instructional capacity as not just a function of the knowledge that teachers have, but as their ability to accomplish new things with that knowledge"(Brown, 2002, p. 70). The notion describes the 'situated interactions' that characterise and influence the design of teaching, suggesting that teachers make constant decisions about how to use materials in the course of practice in light of classroom needs, their own goals, and available resources; and also that curricular materials themselves can provide important support by affording and constraining classroom activity.

PDC as a process of perception and mobilisation of artefacts

Pedagogical design capacity seeks to provide a way of thinking about how teachers perceive and mobilise external artefacts contained in curriculum materials in the process of crafting classroom episodes. As Brown (2009) claims:

> the manner in which teachers perceive and employ particular features of the curriculum design is inextricably bound up with the extent to which they recognise, understand, and are able to teach its content (p.64).

Brown likens PDC to the notion of pedagogical content knowledge (PCK) (Shulman, 1986) in that, both PDC and PCK are generative, that is, they can be developed through practice; they are transformative – implying the transformation of existing resources for the purposes of teaching which also lead to the creation of new resources; and they both manifest primarily during classroom activity. However, whereas PCK can be viewed as an element of what teachers have, PDC characterises a process of perception and mobilisation of resources such as PCK, but is not a resource in itself.

This perspective of PDC as a process of perceiving and mobilising resources and not a resource in itself, suggests that PDC may be a function of both the individual's ability to work with artefacts, and of the nature of the artefacts themselves. On one hand, the individual's effectiveness in achieving results may be characterised by a certain capacity to craft such intelligent partnerships with tools, whereas on the other hand, the qualities of the artefacts can have a profound impact on what teachers perceive and how teachers use the designs. In fact, Brown argues that the way in which teachers perceive and mobilise curriculum materials is highly influenced by the nature of the designs themselves, such that teachers' capacity becomes a function of the teacher-tool partnership; where teachers and materials are engaged in the dynamic interplay between them. The implication hence being that materials which teachers use must be quite explicit in ways that they represent the intended concepts, strategies, and even rationales of the designers.

> the decision to adapt or adopt is a function of both the degree of alignment between the teacher's goals and designers' goals, as well as the quality of the materials in representing the necessary concepts, tasks, and purposes of the activity in ways that were intelligible to the teacher (Brown, 2002, p. 75)

Elements of Individual Pedagogical Design Capacity

According to Brown (2002), how teachers transform available resources into meaningful uses in the classroom depends largely on two aspects. Firstly, it depends on teacher's ability to perceive needs and opportunities in their classroom situations. This ability necessitates a competence on the side of the teacher to read such situations. This competence is evidently informed by teacher's existing knowledge resources. Secondly, it depends on teachers' ability to generate and apply appropriate responses to those needs in the light of available personal and external resources. Understandably, variations in the crafted classroom designs are expected dependent on individual teacher's abilities, even with common sets of curricular materials. PDC therefore provides a "revealing" way of accounting for such variations in teachers' implementation of common sets of curricular materials.

For example, teachers with *different* knowledge resources are likely to perceive the affordances in curricular materials in different ways. A teacher who lacks understanding of a particular mathematical concept for instance, is less likely to understand the rationale behind the incorporation of such a concept in a textbook chapter. This impedes their capability to envision possible applications of that concept in their teaching. On the other hand, a teacher with well-developed understandings of such concepts might be more likely to grasp the author's intentions and get the big picture. This would enable them greater capacity to use the designs in more pedagogically meaningful ways.

In the same way that teachers' resources influence how teachers perceive the affordances in the materials, they also influence the process of mobilisation of the materials. For instance, teachers who lack understanding of the intended strategies may be more likely to rely on procedures in the textbooks, whereas teachers with a stronger grasp of the relevant practices may be more inclined to apply their own improvisations in the classroom.

This section on PDC has highlighted critical issues pertaining to the notion of pedagogical design capacity that help us understand better the teacher-text relationships:

a) PDC points to a particular type of competence which is highly influenced by a teacher's knowledge, but not the same as knowledge expertise.

b) PDC is best characterised as a process rather than an entity, and provides a means of characterising the different ways in which different combinations of teacher resources influence each teacher's capacity to design classroom episodes with curriculum resources. Teachers using the same textbook may achieve very different outcomes given their differences in knowledge and their classroom settings, but at the same time, teachers with very similar resources may produce very different classroom episodes depending on how they employ the curriculum resources.

c) PDC highlights in a way the fact that availability of resources to a teacher does not necessarily mean the teacher is able to perceive and mobilise them in ways that support their teaching goals. There is more that goes in the actions of perceiving and mobilising the available resources.

d) Understanding the teacher-text interactions involves comparisons of available resources and comparing classroom outcomes, but as Brown (2002) asserts

> "It is also important to compare the patterns by which teachers ***perceive*** and ***mobilise*** personal and curricular resources in the ***design of instructional settings.*** In this way of understanding the teacher-tool interactions, the goal is not only to identify available resources, but also to identify patterns by which practitioners identify, select, and use such resources…. the patterns of teachers' perception and use will vary in meaningful ways across cases, providing a common framework for understanding patterns in how teachers use curriculum materials"(p.83)

I concur that knowing such patterns brings us closer to understanding the actions of teachers in terms of how they appropriate existing curricular resources to craft their own teaching episodes for their particular classrooms.

CONCLUSION

In concluding this article, I would like to return to the two teachers at the beginning of this article. After outlining all aspects of relationships that teachers must forge with their materials in this article, I would like to suggest that one possible reason for the first teacher to skip some questions from the textbook could have been that the teacher thought about his students and what was possible for them to answer from the exercises. If this was the case, then I would say that was an illustration of some degree of the teacher's PDC. However, at the time of observation it seemed like the decision was a spur of the moment one. And if it was, then the teacher's goals were not aligned to those of the textbook at that point, and this becomes an example of a teacher needing support for the development of the PDC.

The second teacher actually improvised the materials in the textbook by bringing different formulae to the classroom from that in the textbook that both the teacher and students were using. However, the decision to improvise the materials did not seem to have been planned. It also seemed like a spur of the moment decision taken because the reason for introducing other books and new formula was that it was taking students too long to finish the exercise. There must have been a reason why the author of the chapter advanced that particular formula instead of the others. In terms of pedagogical design capacity, I would say that the second teacher also displays low degrees of PDC. However, as I indicated at the beginning of this article, all that I am saying about the two teachers is based on only one observation per teacher, and may not even be how the teachers actually work with their textbooks generally.

This article therefore suggests that in order to understand why the two teachers interacted with their textbooks the way they did, requires for us to know much more than just their knowledge, skills and commitments, but to understand what capacities to perceive and mobilise the textbook to achieve desired goals the teachers have. In other words, the key to understanding the actions of these teachers lies mostly in their pedagogical design capacities.

Pedagogical design capacity addresses important aspects of using the textbook that go beyond teachers' personal resources. It alerts us to pay attention to the design work of teaching and being able to determine the types of designs teachers craft for their specific classroom needs, in collaboration with available curricular resources. As has been stipulated in this article, understanding what teachers do with their textbooks and other curricular resources requires much more than knowing what personal and curriculum resources enter into the partnership. It needs for us to understand what teachers also do with the available resources.

Finally, I need to declare that all of what I am saying is pure speculation as I have no empirical data to authenticate it. That will be done through my study which will unpack this through indepth work with teachers on their relationships with textbooks.

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THE CONSTITUTION OF THE LEGITIMATE TEXT IN THE ASSESSMENT TEXTS FOR THE TOPIC OF NUMBER PATTERNS

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This paper forms part of a larger study which investigated the relationship between what is valued around patterns and what I refer to technically as the legitimate text (Bernstein, 2000) in the curriculum document, in the assessment and in the teachers practice. In this paper I discuss the theoretical tools that I have used to analyse across all these with a focus on the assessment and what is illuminated by the assessment. The wider study shows discontinuities in what is presented as the legitimate text across curriculum, assessment, textbooks and classroom enactments. Four examples from the official National Senior Certificate (NSC) examinations for the topic of number pattern are discussed to illustrate how the tools were used, and what the analysis showed.

INTRODUCTION

Number pattern in the South African official national assessment (National Senior Certificate Examinations) is a highly valued topic and occupies at least three questions with about 25 marks or more out of the total of 150 marks. Zazkis and Liljedahl (2002) have described this topic of number patterns as the heart and soul of mathematics; and Driscoll (1999) and Vermeulen (2007) describe this topic as the proper bridge for early grades between arithmetic and algebra. Number pattern in the current curriculum (DoE, 2003), National Curriculum Statements (NCS), is a topic that starts in Grade 1 and stays in the curriculum until Grade12. The significance of number pattern in the curriculum is reinforced by a great deal of research in mathematics education, where mathematics in general has been described as the study of patterns (Mason, Graham, & Johnston-Wilder, 2008); and where observing patterns in algebra is seen as a natural tendency for some people and is one approach that is used to address difficulties in mathematics because expressing generality is seen as an indication of understanding (Kieran, 2007; Watson, 2009). Number pattern has as one of its key mathematical practices the process of generalization; this process underpins the whole essence of learning algebra in particular and mathematics in general (Mason et al., 2008). Literature on number patterns shows that there are more studies on this topic conducted with young learners (from primary to grade 9) ((Ellis, 2007; Radford, 2010; Rivera, 2010; Warren & Cooper, 2008) and very little in grade 12 (Carlsen, 2010) and nothing in between for grade 10 and 11, because of this gap, there was a need to focus on the FET.

I looked at the Grade 12 assessments because of their high status in the national landscape as gatekeepers and carriers of the legitimate text. The status ascribed to these examinations is seen in the way teachers look upon them and use the examination papers/assessment texts. Assessment texts such as examination papers help teachers know what is expected and so how to best educate learners. To prove just how important assessment is, from the results of the assessment, inferences are made about the quality of teaching that learners went through. For teachers, assessment is an indication of which topics are valued most highly and teachers in most schools actually use these assessments at national level as a guide to decide on the amount of time they spend to teach each topic depending on how much it is valued in the assessment. In this way assessment determines much of the work learners will take on and affects the approaches taken by teachers to teach. This shows just how important and crucial a role assessment plays in education in general and particularly mathematics education. Bernstein (2000) argued that the whole purpose of the device (the pedagogic device) is evaluation, and while he refers to evaluation beyond formal assessment practices, most educators are aware of the key role of assessment for their learners. Content from Grade 10 and 11 is assessed in the national Grade12 examinations in South Africa and hence I examined Grade 12 national assessments given their importance as they provide the high school exit certificate. These are ‘high stakes’ examinations and learner performance in these exams determines whether and what future studies the learner would be allowed to pursue. Furthermore, the education department also uses learner performance from these assessments to rank schools.

It goes without saying therefore that the assessments at national level constitute the legitimate forms of knowledge learners are expected to have acquired. However this knowledge comes in different forms and makes different demands on learners in terms of context recognition which is why I chose the four examples that I discuss below. The four examples illuminate the four domains of practice shown in Dowling’s (1998) model in Figure 1 in the discussion that follows. This implies that the four domains of practice should also be part of everyday teaching and learning because they are examined in the NSC examinations.

The first group of Grade 12 learners from this curriculum (NCS) was in 2008 and that is why I started looking from 2008 to 2009 papers, with the larger study focused on and conducted in 2010. In the 2008 examinations a number of new topics were going to be tested for the first time in mathematics and at a national level. Those topics included: transformation geometry, statistics and probability (in paper two). In paper one, number pattern used to be tested as arithmetic and geometric sequences and series in Grade 12 only. The notion of constant differences sequences was not part of the syllabus, whereas now this topic runs from Grade 1 up to Grade 12 in the curriculum. What follows is a discussion on the tools used to identify and discuss the legitimate text.

THEORETICAL FRAMEWORK

The theoretical framework is drawn from Basil Bernstein's (2000) theory of the sociology of knowledge – the pedagogic device. The pedagogic device is a mechanism of describing knowledge in pedagogic contexts. The pedagogic device is composed of a system of rules; the rules are about the social distribution, recontextualisation and evaluation of knowledge. Bernstein (2000) argues that the whole purpose of the pedagogic device is 'condensed in evaluation' (pg. 36). What this means is that successful pedagogy lies in the individuals that are being evaluated being able to produce the 'legitimate text' – 'what counts as valid knowledge' (pg. 28) in the curriculum. The reason why a lot of learners, specifically in South Africa, are not successful in school is because they are unable to produce the legitimate text, and hence it was necessary for me to problematise the notion of the legitimate text and Bernstein's theory gave me the language for describing what I wanted to say. Legitimate text is anything that attracts evaluation and it does not necessarily have to be written; it can be an action or verbal. It is the social evaluation of what counts as correct/acceptable to do, say or write within a particular context. So the term evaluation as it is used by Bernstein (2000) is a technical term, and as already noted, not necessarily referring to an assessment instrument.

The notion of classification is used to talk about the legitimate text across the analysis. Classification according to Bernstein determines how one context differs from another, he continues and says that the classificatory principle is the one that determines the recognition of the context. If the classification is weak, the boundaries are blurred and the context is ambiguous, then recognition is not easy. Alternatively, if the boundaries are clear, then the context is unique and has its own voice, and recognition is easier. How do I then study the constitution of the legitimate text in the examinations? Dowling provides a very useful set of tools for looking at the legitimate text. He has a model that he used to analyse mathematical texts. In his model he provides a way of saying the classification is not only about the content but is also about the form used to express the problem. And so I discuss Dowling.

Dowling (1998) has elaborated further on the notion of classification of mathematical text. He argues that classification of a mathematical text can be seen and described in two ways: that is the content[13] from which the text draws from (it can be weakly or strongly classified) and the form of expression that the text takes (which too can be strongly or weakly classified). This is illustrated in Figure 1 below. Dowling posits that if both the content and the form of expression are specialized then classification is strong, and the domain of practice is esoteric. Conversely, weak content and expression is in the public domain of practice. He completes is 2 X 2 matrix by describing weak classification at the level of content and strong classification at the level of expression as the descriptive domain; while the expressive domain is made of

[13] Dowling uses the word content to refer to context; these two (content and context) are also used interchangeably in his writings.

strong classification at the level of content and weak classification at the level of expression.

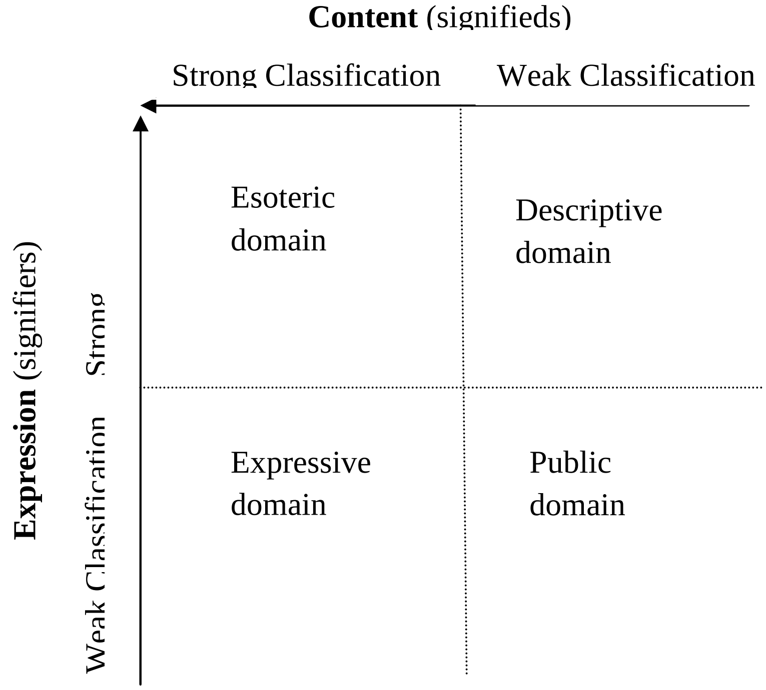


Figure 1: Dowling's (1998: 135) domains of practice

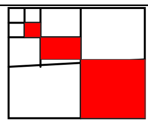
From a study of the literature related to patterns in the school curriculum in the broader study and related curriculum analysis, a framework was developed so that analysis could separate content, process, conventions and context in all the data studied. This framework was used to present the legitimate text in table form from all the sets of data dealt with in the broader study. In this paper I show how the same framework together with Dowling's domains of practice, was used to talk about the legitimate text in relation to assessment.

RESULTS AND USE OF TOOLS

The following table is a summary of the examination papers analysed. As noted above, the framework (content, process, conventions and context) used to categorise examination questions within this table emerged from the literature review and the study of the curriculum. In the table is a brief discussion of the preliminary examination papers and the final examination paper for the year 2008 (DoE, 2008a, 2008b) and 2009 (DoE, 2009a, 2009b).

Table 1: Examination papers

	Prelim (2008)	**Nov. (2008)**	**Prelim (2009)**	**Nov. (2009)**
Content				
1. Linear	Mixture of arithmetic and constant 1; 2; 1; 5;	Mixture of arithmetic and geometric ½; 4; ¼;	Combination of two linear 1; 1; 3; 2; 5; 3;	5; 9; 13; 17; 21; …

	1; 8; 1; 11; …	7; 1/8; 10; …	7; 4; ... $\sum_{n=1}^{50}(2n-1)$	$\sum_{t=0}^{99}(3t-1)$
2. Quadratic	3; 6; 11; 18; 27; …	8; 18; 30; 44; …	1; 5; 11; 19; ...	-3; -2; -3; -6; -11; …
3. Geometric				
- Diverging	Word problem based money for homework done			5; 125; 3125; 78125; 1953125; …
- Converging	$8(x-2)^2$; $4(x-2)^3$; $2(x-2)^4$; …	$8x^2+4x^3+2x^4+...$		Word problem based on growth of a tree
Process	Extend, find general term, find sum, find position, explain and justify		Which term of the sequence is 2549	Justify, prove T/F
Conventions	$8(x-2)^2$; $4(x-2)^3$; $2(x-2)^4$; …	$8x^2+4x^3+2x^4+...$	sigma notation	sigma notation
Contexts	Money, numeric and algebraic contexts used	numeric and algebraic contexts used	Fractal geometry, numeric and algebraic contexts used	Growth of a tree, classroom scenario, numeric and algebraic contexts used

Literature showed that in a mathematical activity, content and the context are acted on and the actions have been classified as mathematical processes. Mathematical conventions are appealed to in the process of performing these mathematical processes. Therefore, for each question from the examinations I separated content, process, conventions and context as shown in the Table above. From this Table I discuss four examples which illustrate each of the categories in Dowling's model; the public, the esoteric, the descriptive and the expressive domain; and so the legitimate text.

Example 1 below is an example of a problem that falls within the esoteric domain of practice.

<u>Example 1: Esoteric domain of practice</u>

Consider the following sequence: 3; 6; 11; 18; 27; ... 4.1 Determine the 6^{th} and 7^{th} terms of the given sequence, if the sequence behaves consistently. (2)

4.2 Determine a formula for the general term, ***p***, of the sequence. (4)
4.3 Use your formula to calculate ***p*** if the p^{th} term in the sequence is 627. (4)

(Taken from the DoE/Preparatory Examination 2008, question 4 page 4 of Paper 1)

The content is explicitly written as 3; 6; 11; 18; 27 ... This is a quadratic number sequence. The first process learners have to go through is extending the sequence by finding the next two terms of the sequence. The next process is to generate the algebraic expression that represents the pattern in terms of ***p***. The last process is solving for ***p*** and in the process learners have to know that ***p*** is a positive number. ***p^{th} term*** is a convention that is used to denote the general term in this topic. The context from which this problem draws is specialised mathematical content and the form used to express the problem is specialised and mathematical. The use of 6^{th} and 7^{th}, the use of ***p*** to denote natural numbers and ***p^{th} term*** are forms of expression used in mathematics. The demands made by this kind of a problem are fairly straight forward if a learner understands the symbol system and what it means. This shows that when the domain of practice is esoteric, then the demands made by the problem are relatively low because classification is strong at both levels that is content and form of expression. By demands, I am referring to the processing load (Warren, 2000) made by the problem because of the non-academic problem contexts drawn on or used to express the problem. Therefore an esoteric domain of practice is unambiguous and has gains for learners because it opens access to the recognition rule of both the context and form of expression.

What follows is an example of a problem that falls within the public domain of practice.

Example 2: Public domain of practice

5.1 Kopano wants to buy soccer boots costing R800, but he only has R290, 00. Kopano's uncle Stephen challenges him to do well in his homework for a reward. Uncle Stephen agrees to reward him with 50c on the first day he does well in his homework, R1 on the second day, R2 on the third day, and so on for 10 days.
5.1.1 Determine the total amount uncle Stephen gives Kopano for 10 days of homework well done. (5)
5.1.2 Is it worth Kopano's time to accept his uncle's challenge? Substantiate your answer. (2)

(Taken from the DoE/NCS Preparatory Examination 2008 Question 5 of Paper 1 page 4)

The content of this problem is a diverging geometric sequence with a common ratio of 2. Learners have to process the sequence from the word problem. Processes that learners have to go through are expressing the sequence, finding the sum of the first ten terms of the series and justify their answers by arguing if the offer should be accepted. There are no conventions used in the expression of the problem. From the story learners have to generate the sequence and recognise that it is a geometric diverging sequence. The first question if it was reading as: find the sum for the first ten terms, then the domain would be esoteric; but the question says: ***Determine the total amount uncle gives Kopano for 10 days of homework well done***. So both form of expression and context are non-mathematical and therefore this problems falls within the public domain of practice. The context could be thought of as "incentive based performance" and there are no mathematical symbols and conventions used to express the problem. The demands made by this problem at the level of context recognition are high; the processing load is increased - classification is weak.

Example 3 below is an example from the descriptive domain of practice.

Example 3: Descriptive domain of practice

Question 5

Data regaridng the growth of a certain tree has shown that the tree grows to a height of 150 cm after one year. The data further reveals that during the next year, the height increased by 18 cm. In each successive year , the height increases by $\frac{8}{9}$ of the previous year's increase in height. The table below is a summary of the growth of the tree up to the end of the fourth year.

	First year	Second year	Third year	Fourth year
Tree height (cm)	150	168	184	$198\frac{2}{9}$
Growth (cm)		18	16	$14\frac{2}{9}$

5.1 Determine the increase in the height of the tree during the seventeenth year. (2)

5.2 Calculate the height of the tree after 10 years (3)

5.3 Show that the tree will never reach a height of more than 312 cm (3)

(Taken from the DoE/NCS November 2009 Question 5 of Paper 1 page 4)

Presented here is a table with data that shows the height of the tree over four years. Differences between successive heights are found and labelled growth in centimetres. The content presented in the table is a geometric converging series. So the first two questions require learners to continue the sequence. This is a process known as near generalisation (Driscoll, 1999) within mathematics education literature. And the last question requires learners to find the sum. In the process of finding the sum learners have to notice that this is a converging series and chose the correct formulae. In the phrasing of the problem there are no mathematical symbols that are used, but a table is used to display the data. The context is a non-mathematical context based on growth of a tree, and the broader context within this could be gardening or forestry; therefore classification at the level of context is weak. However classification at the level of expression is mathematical because of the table that is used to display the data, so classification at the level of form of expression is strong. The demands made by this problem are very high especially if one is not familiar with the context of how trees grow. However, the demands for recognition made by this problem are not high because some of the numeric data are presented in table form.

Next I discuss example 4 from the expressive domain.

Example 4: Expressive domain of practice

> Question 2
> 2.2 Nomsa generates a sequence which is both arithmetic and geometric. The first term of the sequence is 1. She claims that there is only one such sequence. Is that correct? Show ALL your workings to justify your answer. (5)
>
> (Taken from DoE/November 2009 page 3, question 2.2 of Paper 1)

The content here is a sequence which is both arithmetic and geometric and the first term is one. The processes involved are generating the sequence from the information given and proving that the generated sequence is the one Nomsa is referring to. There are no conventions used to phrase the problem. I have categorised this problem as expressive because the context is strongly classified, it is a purely specialised mathematical context that is used to generate the problem, while the form of expression is weakly classified. There are no mathematical symbols and notation used in the expression of the problem. The demands made by this problem are very high; it requires learners to do a lot of processing. For example learners have to know that the sequence has a difference of zero for it to be arithmetic and a common ratio of 1 for the same sequence to be geometric, and if this is the case then the sequence is a constant sequence of 1's (1; 1; 1; 1; ...). To communicate all this, a learner must be able to write using the language of expression – the symbol system of algebra.

The criteria I used to say the demands are high or low depends on the context and the form of expression used to formulate the problem, if the context is non-mathematical then the processing load is increased thus making the demands high at the level of recognition. If the expression is non-mathematical the processing load is still increased and so the demands are high. Consequently the esoteric domain is the one with relatively low demands because it has neither of these conditions.

From the four problems, two have weakly classified contexts that are the descriptive and the public domains of practice and the demands made by both were high. The expressive had a strongly classified mathematical context with weak classification at the level of expression and the demands made by the problem were high. So we see that the legitimate text in the assessment texts takes on different domains of practice and obviously this has serious implications for teaching and learning. I end this paper by discussing these results and the implications they have for learning and teaching.

CONCLUSION AND FINDINGS

Figure 2 below shows the results from the broader study (Luxomo, 2011), which I have not illustrated here because all I am trying to do is show what is illuminated by the assessment when using Dowling's domains of practice as a tool. Figure 2 shows that the domain of practice engaged by the curriculum is the same as the one preferred by the teacher observed in the larger study. It also shows that the textbook that was used in the classroom observed offered problems which fall under the esoteric and the descriptive domain of practice only for this topic of number patterns. As we can see below, and from the four examples discussed earlier, it is clear that the assessment draws on two more domains of practice, which were not drawn on in the textbook or in the teacher's practice. This has serious implications for teaching and learning, because both the textbook and the curriculum are resources that a teacher draws from, and they contain the legitimate text. Clearly the legitimate text constituted in these is limited compared to the one constituted in the assessment texts. The legitimate text is also limited for a teacher who only has these two (curriculum and textbook) as resources. This means that teachers should be aware of the resources they have and how the legitimate text is constituted in each. Teaching and learning should also draw from all domains discussed in this paper if learners are to have access to recognition and realisation rules which enable them to produce the legitimate text in the assessment. There needs to be greater alignment between curriculum document, the textbook and the assessment. The multi-voice or the noise these documents currently create for the teacher actually puts the teacher in a weak position. It makes work for the teacher trying to read, understand and interpret across these. This suggests that when there are so many documents (including those not analysed in this study) to read and interpret, the teacher will probably align with the curriculum document anyway.

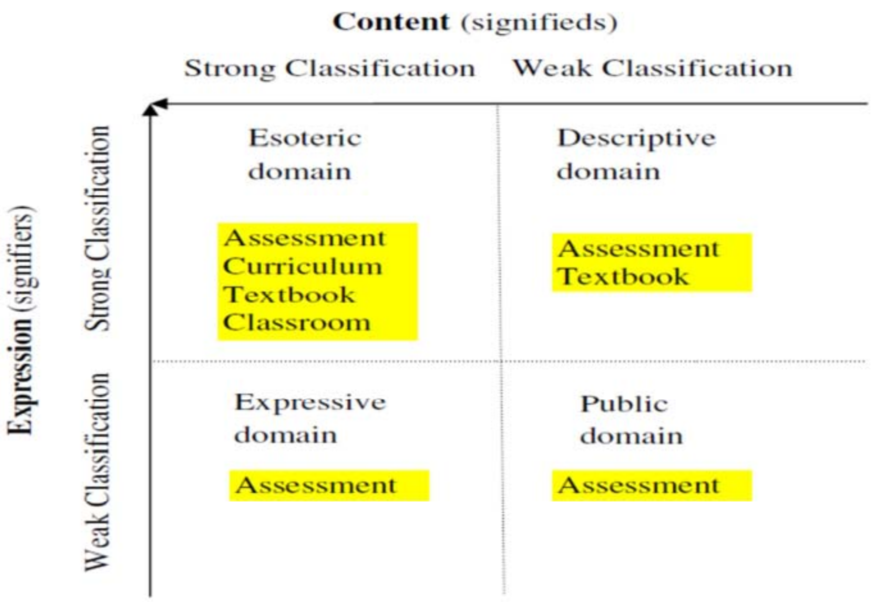


Figure 2: Summary of findings using Dowling's model

Acknowledgements

This study was funded by the Sasol Inzalo Foundation. Any opinions, findings and conclusions expressed are those of the author and do not necessarily reflect the views of the Inzalo Foundation. I would like to also acknowledge the helpful input I received from Reviewer 1.

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What should be which is not: Case study of African students mathematics experiences in a former White School

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African students' mathematics learning has been regressing instead of improving in South Africa. A case study of two fifth grade classrooms in a former White school in the Eastern Cape describes experiences of African students in this new context. Reforms in the South African government had open doors to African students to study in any school in South Africa. Big numbers of students moved to the previously white schools to seek quality education. This case study revealed that the quality of education received by African students is characterized by overcrowded classrooms, passive learning, lack of quality mathematics instruction, outsider status of the students, and action-based macro-aggressions.

INTRODUCTION

Multilingual students portray South African classrooms dominated by African students. However, this diversity is not well represented amongst teachers in former White schools of South Africa. There is rich literature from empirical research on multilingual mathematics classrooms that advocate for utilization of code-switching as a tool to enrich mathematical discourse in classrooms (Adler & Setati, 2001). Setati (1998, 2002), Adler (1998), Vithal (2003), and Feza and Webb (2005) revealed that South African, African students use their native languages to acquire conceptual understanding of their mathematical ideas. However, most of the South African research on multilingualism had focused on African students learning in their cultural context; hence the teachers teaching them are also multilingual. Very little has been done in understanding how these African students learn mathematics in former White schools. This paper aims to describe fifth grade African students' experiences in a former White school.

THEORETICAL FRAMEWORK

English proficiency plays an important role in learning mathematics when the language of instruction is English. Khisty and Morales (2004) highlighted that second language students do not go through similar processes, development, and factors when they learn and talk mathematics. Hence, Setati and Adler indicated the role played by code-switching in enhancing mathematical discourse. The same treatment given to these students and English First Language peers become a marginalizing treatment.

Carignan et al. (2005) observed that a requirement to enroll African students in a former White school of the Eastern Cape was commitment to speak English both at school and at home with family. A message of "become us" instead of being you.

Mathematical understanding develops better when students ask questions and challenge their own thoughts (Nicol, 2005) and it dominates successful multicultural pedagogy instruction. This mathematical discourse involves language of instruction; therefore, whose language should be used for this discourse gives power and success to those students who speak the language. Matang (2006) elaborated the need for integrating the language of the learner into mathematical instruction.

South African history of education is embedded on students' learning and access to quality education. This history cannot be ignored in studying students' experiences. Hence, this study employs Critical Race Theory to elicit these students' experiences. Racism has been used as a powerful tool for discrimination and its' practices are innate and unintentional. However, their impact on the oppressed are damaging as they demoralize and lead to self condemnation (Landson-Billing and Tate, 1995). Racism is a neglected variable in education literature. Multicultural and gender studies cannot encompass racism in addressing injustice in education, however it has grave impact in enhancing injustice. The history of racism in South African education is highly demonstrated by performance trends in mathematics education. Over and over again, mathematics performance in South Africa from TIMMS, SACMEQ, and the National Senior Certificate indicates hierarchical performance, with African students at the bottom end followed by Coloureds then Indians, the White students are always on top. Critical Race Theory was employed for the strength it has to extract racism embedded in practice, communication and in actions. Racism is a cognitive habit of entitlement imposed consciously or unconsciously by the dominant group to the subordinate group (Landson-Billing and Tate, 1995). The intrinsic beliefs that one race is superior over all others and has the right to supremacy Lorde (1992) cited (Solózano, 2000), are not erased through elections or through change of constitution. They demand emancipation of the oppressed who will then take actions against any form of discrimination. Yet, racism is less theorized, in education because of the "traditional claims of the legal systems to objectivity, meritocracy, color-blindness, race neutrality, and equal opportunity" (Solórzano, 1997, 6).

METHODOLOGY

Research Site

The data was collected from a former white school in the Eastern Cape of South Africa. Below is a vignette that draws a picture of the school.

A tall building built in 1894 for Afrikaners' students in a small town in the heart of the Eastern. The Afrikaans influence is still reflected by the majority of teachers in the school that are native speakers of Afrikaans. However, the medium of instruction is English. The front of the building has three offices, the secretary office with a computer; fax machine, a photocopying machine and a telephone tiled with white ceramic tiles on the floor. Next to it is a staffroom with shiny wooden floors and neatly organized utensils. Adjacent to the staffroom there is a passage leading to the principal's office, a little further to the staffroom. In front of the principal's office there are beautiful golden trophies, inside are comfortable leather chairs surrounding a long table and a principal's chair on the other side of the table. Next to the secretary's office there is a security door towards the classrooms operated by the secretary. The passage from the security door is also tiled with ceramic tiles. Behind this administration building are overcrowded, noisy classrooms with wooden floors that show thirst symbolized by the amount of dust piling on them. The color of the wooden floor became white. The electric lamps that are hanging have a dust that turned black in color. The school grounds have lots of tiny papers that fill up some of the water drains.

Participants

A high school qualified mathematics teacher, in his middle age with two fifth grade classrooms that have 100% African students were observed. The teacher had 20 years of high school teaching of mathematics to white students, and a native Afrikaans speaker. Elaborating on his experience the teacher stated "It is my first year teaching in a school dominated with African students". He was the only mathematics teacher for 4th grade to 9th grade. During the time of the observations he was in his 7th month in the school during the last term of the year.

Data Collection

Observations

Teachers interpret development of conceptual understanding of students differently. This study seeks to make sense of these interpretations through passive and participatory observation eliciting students' participation and instruction received.

Interviews

Follow-up informal interviews after each observation were conducted with the teacher to understanding the lesson observed confirm observations with the teacher's voice (Ely et al., 1991).

Classroom artifacts

Classroom artifacts consisting of worksheets, classroom photos, teaching and learning aids were collected to triangulate with the observations and the interview notes.

Data analysis

Data analysis was inseparable with data collection in this study to inform continuous data collection and to engage with data. Analytical memos assisted in making dialogue and making sense of the data (Ely et al., 1991). The ongoing data analysis involved three stages of analysis i.e., descriptive, thematic and analytical.

Descriptive analysis

The transcribed verbatim from interviews, observation field notes were typed and numbered in rows. They were annotated using low inference phrases describing each line of the typed verbatim and then putting those phrases on a table. The same approach was used for artifacts. Each data source table was color coded to look for similarities, differences and odd occurrences. Analytical memos were used to describe emerging pictures from each table. The emerging patterns were tabled as descriptive codes.

Thematic Analysis

The table of descriptive codes was then color coded with the aim of finding thematic codes across three data sources (Miles & Huberman, 1994). The table became more complicated with the observation column too much longer than the others. The data was left unattended for two days. During the third day Visual Basics were used. The integration of the tables with photos of the school from the research log gave a clearer picture. The connections and themes started to emerge. The themes were then triangulated with literature on Critical Race theory adapted by Ladson-Billing and Tate (1995) from the U.S law.

RESULTS

What should be which is not

White privilege made African students uninvited company in this former White school. The dichotomous culture demonstrated by the clean, organized, well resourced administration block versus water drains filled with papers, dusty classrooms with torn, depleted flying posters and students with no books reflect the protection of White privilege. African students are the 97% of the school enrolment and their presence in the school is not embraced by the environment they are learning under. They are neither represented in the teaching staff, as the majority of teachers are White with only 1 % of African teachers to teach an African language. Landson-Billing (1995) refers to racism as power over the dominated group that is a cognitive habit of White people. This environment of a well organized administration block with the Principal’s office, staff room and secretary office demonstrated the status of the people who occupied this block versus those who occupied the classrooms. By not giving students the responsibility to take care of their learning materials showed that they had no right to any property. They have been done a favour already by being in the same school; however they will not get similar rights to cleanliness, and belonging. They are complete outsiders. Louw (1996) revealed that Black South Africans entered the democratic South Africa with no property ownership because of apartheid even those who were in black homelands. The right to property ownership gives a lot of power to the property owners leaving those with no property rights with the servant status (Landson-Billings, 1995). In this school teachers had more responsibilities and students were passive members of the school. Teachers carried pencils, class-work books, and worksheets as they moved from class to class.

Students assisted teachers' carrying the loads. However, students did not have books to read nor write to when teachers were not in their classrooms or at home.

This school culture showed the effects of the retraction and dwindling of African property rights by Whites (Mbaku, 1991). Both students and parents do not question this act of not owning learning material because of the self condemnation and demoralization effects of being marginalized (Landson-Billing, 1995). The shiny golden trophies displayed next to the principal's office indicated success of his school dwelling on the past achievements. The past that consists of racism, grouping of races and dominant White access to privilege still prevails in the administration of this school.

Shift of property rights

Bantu education that was the Apartheid baby was implanted with the aim to keep Africans in the lower status than whites through education. Hence, Africans could not receive quality mathematics and science learning in their schools. The African classrooms of the Apartheid era were characterized by overcrowded classrooms, whole class instruction, lack of resources, and unqualified teachers. In this school that is a former White school, the fifth grade classrooms observed were overcrowded. Out of the two fifth grade class one had 45 students in a classroom area of 5 by 6 meters with no navigational space, while the other had 52 students in the same amount of space. Their mathematics teacher is a high school qualified mathematics teacher. Recounting his concern Mr. Wessels (pseudonym) the mathematics teacher conferred that "I was a high school maths teacher, when I came here I realized that there is too much back log". Seemingly these students movement to this White school included Apartheid legacy of African overcrowded classrooms, lack of resources, taught by high school teacher in elementary school. Mr. Wessels (pseudonym), the mathematics teacher described the rationale behind overcrowding of classrooms as follows:

It is about money. The numbers are more important for the Principal to earn more money in his salary. So the students are not important. When I teach this class I feel that I reach only half of the class. All others are neglected. I do not understand how that could be allowed.

This conversation indicated that students were job creators and determinants of salary scales for the principal. As a result of this Mr. Wessels as the school mathematics teachers was experiencing difficulties in achieving mathematics quality teaching. This act indicated a shift of property from the Whites to the government who give incentives to Principals for enrolling more students regardless of space and number of teachers the school has. With this kind of learning environment the teacher was left with limited choices in his teaching and his instruction was dominated by teacher talk and teaching of rules.

Student status

The following extract presents the teacher's daily observed instructional approach: Teacher writes on the chalkboard:

16430

<u>-9846</u>

6584

Teacher: Can I subtract 6 from 0?

Students (in chorus): No

Teacher: So

Students: Borrow

Teacher: Can I subtract 4 from 3?

Students: No

Teacher: So

Students: Borrow

Students: No

The teaching approach practiced in this classroom is mostly teacher talk, with students sitting and responding chorally to the questions. Students' sat in rows facing the chalkboard. Students' status in this instructional method is of an inferior status with no prior experiences or knowledge. Opportunities to connect the known to the unknown are not presented. Creativity, critical thinking and higher order thinking are embossed on by this approach and keep these children in the same era that they were hoping to run away from when they left African schools. Marking of students' work was done during the teaching time leaving some students unattended. In one observation students made a long cue towards the teacher's table with their test books. The teacher marked them one by one while the majority was noisily doing nothing useful. Solórzano (2000) describe macro aggressions as inadvertent and understated forms of racism. Mr. Wessels showed concern about overcrowded classrooms that provide incentives to the Principal showing moral concern to the practice. However, the same individual demonstrated non-commitment and insensible attitude when marking students' work during instructional time and leaving the majority of students unattended.

DISCUSSION

The results of this study are not generalisable but transferable. They describe experiences of African students in this particular setting only. Apartheid legacy prevails in this school and macro aggressions are demonstrated through actions than verbal communication. Macro aggressions are subtle insults of discrimination towards the discriminated group. In this case there were action-based than verbal. White privilege is protected by allowing the school to enroll as many for the benefit of the Principal while overcrowding is manifestation of dis-functioning classrooms (Scherman and Howie, 2008). The government supports this practice by providing incentives to those Principals who enroll more students regardless of space. In keeping their reputation status, White children flew away from this school when African students enrolled hence only 3 % of them are still in the school. Landson-Billing and Tate (1995) revealed that the reason for the White student flight is caused by the fact that these White schools loose reputation when Black students enroll in them. To these schools the presence of black students is "contaminating" hence the right to exclude them is demonstrated through few white students' enrolment if any (p.60).

The physical environment of this school demonstrates the "rights to use and enjoy the privileges of whiteness" (Landson-Billing and Tate, 1995, p.59). Whiteness allows "teachers and administration the extensive use of the school property" enjoying cleanliness, luxury and resources while the African students expose their lungs to dust and are with no books. The cognitive habits, action-based macro aggressions of whiteness demonstrated by the teacher denied students' access to quality education if any could be experienced.

IMPLICATIONS

Critical race theory has potential in bringing forth inequities in education as racism still plays a significant role in South African lives. The hierarchical race performance in education reveals the in-depth of racism as a marginalising factor in education for justice. Unless research and policies confront racism, its' power cannot be easily dealt with as it is a cognitive habit, an inherent right, and a privilege that will always be protected. This study elicits what South Africa would like to claim as something under a watchful eye. Conceptual whiteness and conceptual blackness are ingrained in people's minds and beliefs. Social justice in education will not take place fully until the oppressed are able to embrace self instead of working hard to become whites they will never be. Until they are able to confront macro aggressions, own property, belong, and live instead of surviving there is no justice in the South African system. The students in this study are outsiders in their school; they experience subtle insults of racism every day. This study calls for more studies of this nature that will challenge our education practices for the betterment and eradication of racism.

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STUDENT TEACHERS' ORIENTATIONS TO PARTICIPATION IN A DISCOURSE[14] OF ENGAGING WITH LEARNER MATHEMATICAL THINKING: A FOCUS ON ERROR

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I attempt to describe and analyse final year mathematics education student teachers' orientations to learner errors. This is part of a study seeking to contribute to knowledge of how the notion of engaging with learner mathematical thinking is recognised and focused on in teacher education, and what and how this is realised by student teachers. Student teachers are asked to explain their reasoning of learner responses to tasks designed around algebraic error and comment on how they might deal with these. The descriptions of their own reasoning processes are then analysed. Initial findings reveal that student teachers orientations are more towards absence rather than presence in the learner contrary to the constructivist theory espoused in their training programme. Tentative implications for teaching are then discussed.

INTRODUCTION

In this paper, I describe and analyse student teachers' orientations towards learner errors as they participated in initial focus group discussions which were aimed at establishing their views on learner mathematical thinking. I also use student teachers' explanations as they worked with a task designed around algebraic error. The student teachers were participants in a larger study concerned with twenty final year University student teachers' preparedness to engage with the discourse of learner mathematical thinking. This investigation was carried out in two ways. The one way was by exploring the nature of the discourse in the pre-service mathematics education curriculum through document analysis of course outlines and assessment materials, and interviews with mathematics education teacher educators and student teachers. The other way was by exploring student teachers' positioning, recognition and participation by involving them in task-based interviews designed around learner mathematical thinking.

These activities included student teachers solving algebraic problems and then anticipating learner difficulties; working with algebraic learner errors elicited from literature as well as those elicited from learners' own working. They also had discussions around a 5-minute video clip of a teacher introducing algebra to a grade 8 class. Moreover, they discussed how they would teach roots of a quadratic equation

[14] I am using Sfard's (2007) and Gee's (2005) notion of discourse from a commognitive and sociolinguistic stand point to mean different types of communication that bring some people together while excluding others. These include a combination of speaking or writing, doing, being, or believing.

to learners experiencing difficulties. Finally, student teachers were also asked to write journals based on the identified readings to establish how they thought they were informed pertaining to learner thinking. The choice of the participants was purely purposive from a total number of twenty seven student teachers at fourth year level in the mathematics education programme. The understanding was that since the student teachers[15] were nearing the end of their mathematics teacher education programme, they should ably interact and bring to the fore their thinking of engaging with learner mathematical thinking. How this aspect is dealt with in mathematics teacher education forms part of my analysis in the main study. From the initial analysis conducted so far, I can infer that there is no particular topic called “errors and misconceptions” in the outlined topics of the curriculum. All the four teacher educators interviewed indicated how they deal with it when focusing on particular topics such as “problem solving” and “school mathematics”. In Bernstein’s (1982, 1996, 2000) terms this is an indication that engaging with learner mathematical thinking has no identity and voice of its own, and hence implicit in the mathematics teacher education curriculum.

The purpose in this paper is to share with you an attempt towards developing adequate descriptive and analytic resources to talk about what student teachers view as engaging with learner errors, and how they do so in terms of the reasoning processes they bring forth. I assume that student teachers’ legitimation of engaging with learner mathematical thinking illuminates at least some of the criteria that are transmitted by their teacher educators in their lectures even if only implicitly, hence what it is that is privileged. This is based on three propositions, firstly that any pedagogic practice whether implicit or explicit is highly evaluative in that it transmits criteria for the legitimation of text (Bernstein, 2000). Secondly that engaging with error analysis devoid of a theory of learning is rendering the task ineffective (Olivier, 1989; Peng & Luo, 2009; Sfard, 2008). Thirdly that engaging with learner errors is desired in teacher education in terms of providing opportunity for student teachers to develop deeper understanding of the mathematics they are to teach as well as enhancing their understanding of learner mathematical thinking (Even & Tirosh, 2002).

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[15] I have used “student teachers” to refer to both in-service and pre-service teachers taking into cognizance that these could be two distinct groups by virtue of their teaching experience especially that most PCK development is enabled in the classroom. My aim in the study was not to compare how different their responses to learner mathematical thinking would be but that despite their experiences what and how do they talk about learner mathematical thinking especially that both groups attended the same Mathematics Education courses. The in-service teachers had a secondary teachers’ diploma in mathematics education with at least 5 years teaching experience and were upgrading to degree level.

LEARNER MATHEMATICAL THINKING AND TEACHERS' ENGAGEMENT

Learner mathematical thinking and teachers engagement with this has been widely researched in the field of mathematics education. Some focus on the nature of errors (Borasi, 1987; Nesher, 1987; Olivier, 1989; Ryan & Williams, 2007; Smith, diSessa, & Roschelle, 1993) and others focus on strategies for dealing with these (Jacobs, Lamb, & Philipp, 2010; Peng & Luo, 2009). Even and Tirosh (2002) focus more directly on what it means for a teacher to be aware of and knowledgeable about learner mathematical thinking, asserting that this impacts positively on the practice of teaching. They talk about the nature and strategies of learner thinking, and go further to conceptualise the kind of environment that is required if teachers are to access learner mathematical thinking. Even and Tirosh (2002, p. 229) encourage a classroom culture where learners are to make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true. More specifically they argue that a teacher's awareness of student thinking is multi-dimensional. It extends from being aware of learner conceptions, to awareness of developing in the learner both instrumental and relational understanding, and to awareness of a classroom culture that supports such. They assert that while these dimensions are analytically distinct, the one always involves the other.

That dealing with learner thinking is an important part of teachers' professional knowledge has long been recognised. Shulman's (1986, p. 9) seminal paper put this into the foreground as part of PCK and emphasized on teachers having "an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons." Shulman states that in the event that these conceptions and preconceptions turn out to be misconceptions, teachers should be knowledgeable of the strategies that would help in reorganizing the understanding of learners. It is interesting that despite a great deal of research following Shulman's work little has focused on the actual engagement of this at the pre-service level.

DISTINCTIONS AND THEORETICAL ORIENTATION

In the field of mathematics education, research has distinguished between slips and errors. Errors are systematic. For constructivists, the source of errors is misconceptions. I begin with a discussion on slips, and then discuss errors from a constructivist perspective.

Slips are wrong answers that are sporadically or carelessly made by both experts and novices; they are easily detected and are spontaneously corrected Olivier (1989, p. 3). Ryan and Williams (2007) talk of 'slips' or 'uncertain diagnosis' in that they are mistakes with no obvious developmental conceptual explanation. They argue that slips can be a result of misreading, misremembering facts, suffering from 'cognitive overload' or jumping to conclusions. I will not focus on slips in the rest of this paper.

ERRORS FROM A CONSTRUCTIVIST PERSPECTIVE

Constructivists view errors as systematic wrong answers because they are applied regularly in the same circumstances and are symptoms of the underlying conceptual structures (Olivier, 1989, p. 3). This means that misconceptions form part of learner's conceptual structures which interact with new concepts, and influence new learning, mostly in a negative way: misconceptions generate errors (Olivier, 1989, p. 3). Moreover, errors are superficial behavioural results of actions performed on a task and that correcting the error disregarding reason for the error committed is detrimental to a leaner's intellectual mathematical development (Ryan & Williams, 2007). In Ryan and Williams' (2007) terms systematic errors result from 'conceptual limitations' or 'identifiable misconceptions' and that diagnosis of such errors are linked to modelling, prototype, overgeneralization, and process-object reification. Modelling error occurs when learners model a situation contrary to the rules of the mathematical game in the academic context of school. An error is diagnostic of prototypical thinking if it is as a result of a culturally 'typical example' of the concept. Error as a result of overgeneralization arises when generalisations that make sense to a set of cases are inappropriately extended (Ryan & Williams, 2007). This can be as a result of what Lima and Tall (2008, p. 6) term "met-before" and "met-after" in that old experiences can affect new learning and similarly new learning can affect the remembering of previous knowledge (Lima & Tall, 2007). When learners fail to navigate between process and object in the learning of mathematics then the error is as a result of lack of completion of the process-object reification (Ryan & Williams, 2007, p. 25).

Constructivists view prior knowledge as the primary source for acquiring new knowledge, hence viewing error as a natural stage in knowledge construction (Hatano, 1996; Nesher, 1987; Peng & Luo, 2009; Smith, et al., 1993). For example, when confronted with new knowledge, learners' prior knowledge can constrain a range of possible operations and answers as well as their understanding of mathematical entities, such as how to formulate a given problem mathematically (Hatano, 1996, p. 209). This process results in learners creating misconceptions which from the constructivist perspective are viewed as part of learning. And prior knowledge, though constraining, plays a foundational role in developing understanding (Smith, et al., 1993). The teacher should be able to synthesize an accurate picture or model of learners' misconceptions from the evidence inherent in their errors. Thus the teacher should be able to identify, interpret, evaluate and remediate learner error (Peng & Luo, 2007). The role of the teacher is therefore central in helping learners construct new knowledge by being aware of previous knowledge. As the teacher verbalises the target knowledge or when ensuring that learners' understanding approximates expert understanding the learner must partially construct knowledge in the teaching and learning process (Hatano, 1996). This transmitted knowledge becomes useful to problem- solving only when it is reconstructed, that is interpreted, enriched and connected to prior knowledge of the learner. I would like to state here that constructivism as a theory of learning has dominated research on error in recent years.

In the table that follows I exemplify errors from a constructivist perspective.

Error	**Possible constructivists description**
Modelling error: 6 ÷ ½ = 3 or "division makes smaller"	"Division makes smaller" is only appropriate for whole numbers where the numerator is bigger than the denominator. This could also serve as an overgeneralization.
Prototypical error: Failing to identify that a square or a rectangle with different orientations is a rectangle	As a result of a culturally 'typical example' of the concept in that a rectangle lies flat with its longest side horizontal.
Error as a result of overgeneralisation: 'Multiplication always makes bigger'	It is appropriate for whole numbers and it becomes an overgeneralization when applied to all rational numbers, thus including proper fractions or decimal fractions between 0 and 1.
A learner saying 548 is the answer in the problem □ – 1452 = 2000'	Inadequate 'process' conception of the equal sign

Table 1: Errors and possible constructivist descriptions

METHODOLOGY, DATA AND ANALYSIS

From a grounded analysis of student teachers' talk, I make inferences on what counts for them as engaging with learner mathematical thinking, and relate this to Bernstein's (2000) theory of "framing" in the classroom, as developed by Morgan, Tsatsaroni and Lerman (2002) in trying to understand teachers' assessment practices and positionings. They developed Bernstein's distinction between strong and weak framing in that strong framing is an orientation to what is absent in learners' actions on mathematical productions or mathematical thinking. This means that control is with the teacher in this case over what is transmitted, which means strong framing could be linked with a traditional pedagogy. This is so because learners are judged according to what they cannot do. Weak framing is an orientation to what is present in learners' thinking, and hence the learner having control over what is transmitted. The teacher's role is to work with what learners have offered. This is what Bernstein refers to as a liberal progressive pedagogy, in that it involves more knowledge of learners, and knowledge of psychology.

The constructivist perspective in relation to Bernstein's theory so far discussed can be linked to a liberal progressive because what is emphasized is the importance of how the learner thinks. This means that the focus is on what is there rather than what is missing. So what are the learners saying and how does the teacher work with what they are saying rather than focusing on what they are not saying. In Bernstein's terms this could be referred to as weak framing because it is weak in relation to the discipline and it is strong in relation to the learner. So from student teachers' talk is there orientation towards the learners or towards the mathematics, and how is this legitimated especially that constructivism is the theory espoused in their mathematics education courses? Student teachers' initial expressed views especially "the need to evaluate learners' answers" and the "algebraic expression-equation" problem are used to illuminate student teachers' orientations to learner error. This is aimed at inferring what is transmitted as criteria in the mathematics education courses by focusing on what is legitimated by student teachers when inserted in a discourse of engaging with learner mathematical thinking.

DATA INTERPRETATIONS

In what follows, I describe and analyze student teachers' orientations to engaging with learner mathematical thinking using selected excerpts from data as already pointed out above.

Engaging with learner mathematical thinking means evaluating

A major view of the student teachers is that to competently engage with learner mathematical thinking, it is important that they evaluate learners' answers. This became a clear focus in the focus group discussions, and their talk is exemplified in the quotation below.

I think the only way we can engage with our learners in the way they construct their knowledge is, is by evaluating their answers. What I mean is, let us not have this tendency of discarding the learners' answers, immediately the answer is given to some problem we are not out rightly supposed to say this is wrong, ok, but all we are supposed to say is, ask the learner how he arrived at that answer, what motivated the learner to arrive at that answer, look at the procedure and then try as much as possible to reason with that learner and then direct them in the correct procedures. So the ideas there is not discarding the answers that they give us to certain problems but then trying to rehearse with them even those procedures, rehearse with them, where do you see some, where do you raise questions even in the procedures, it could be they started so well, now at some point they missed it. So how well can you direct them, so let us try to be with them through their procedures and try to rectify their procedures, not just bringing things to them that this is the way it should be done, yours is not correct. Let us try to rectify what they have already done. That is the only way we can motivate them even to fully participate. (Focus group 1)

By evaluating, the student teachers point to a mistake in the procedure; probing learners on how they arrived at the mistake identified; and then correcting the mistake as a way of motivating learners. "Evaluating learners' responses" for them means judging whether the answer is right or wrong in the first instance and if it is wrong, fixing it. From this, I infer that student teachers' orientation is towards absence through orientation towards mathematical procedures. Although when they refer to probing, there is an inclination of orientation towards present.

What is interesting is that the student teachers are located in a constructivist environment – at least at the level of the intended curriculum. The teacher educators in the programme all spoke of the programme being underpinned by a constructivist view of learning. While the student teachers use aspects of constructivist discourse (e.g. engage learners) they are uncomfortable with this. The discomfort is visible in their discourse when they say that engaging with learner mathematical thinking is something they are *supposed to do*. In other words, the student teachers' orientations to engaging with learner mathematical thinking are not coherent with nor comprehensively related to a constructivist perspective. For example, the student teachers do not talk about or distinguish between the **observed error, and explanations as** to **how the error occurred** to **why the error occurred**. From a constructivist perspective, it is only after such a comprehensive analysis of learner mathematical thinking that decisions about remediation can be made (Borasi, 1987; Jacobs, et al., 2010; Peng & Luo, 2009). Jacobs, Lamb & Philipp (2010, p. 173) argue that analysis of learner mathematical thinking *happen in the background, almost simultaneously, as if constituting a single, integrated teaching move,* before a teacher responds. And in responding, the teacher can use errors as *springboards for inquiry* (Borasi, 1987, p. 2). This means providing opportunity for learners to experience the process of error analysis by asking them exploratory questions that can stimulate their erroneous strategies, including questioning whether there are cases where their strategies would be considered correct. Therefore, indicating that errors are not only diagnostic and remediating but also illuminate strengths and weaknesses of available strategies. These skills required to engage with learner error could only be talked about by student teachers as they expressed their views on what is required by the teacher and when they worked with algebraic tasks. Seeing how they demonstrate these skills would require them being observed in the classroom as they teach, which is a different context altogether with its own effects.

In the section that follows I interpret and explain the student teachers' orientations when they engage with a scenario that provokes discussion of learner mathematical thinking. But before I do that, I describe possible explanations of the scenario from the constructivist perspective.

Identifying that learners do not distinguish between expressions and equations

The scenario read as follows:

> A Grade 8 teacher noticed that when learners were asked to simplify the expression
>
> 2x + 5 + 3x – 7 they did the following:
>
> $2x + 5 + 3x – 7 = 0$
>
> $5x - 2 = 0$
>
> $5x = 2$
>
> $x = \frac{2}{5}$
>
> Source: Adapted from Wagner and Parker, 1993

How would constructivists interpret this scenario?

From the constructivist perspective, the scenario can be explained as a result of a tendency by learners to overgeneralize and in this case overgeneralization of new learning (i.e. solving equations) over remembering previous learning (i.e. simplifying expressions). This is so because what you meet afterwards, you tend to overgeneralize backwards, met-after according to Lima and Tall (2008). The scenario can also be explained operationally in that the learners see the expression as an operational activity (Sfard & Linchevski, 1994). So they want to make things equal something, they want to close stuff off, which is early closure – also similar to $2+3x=5x$ as reported in the CSMS study; we know that. The error might also be explained in terms of the learner not having attained an object conception of equation (parallel to Sfard's discursive theory – non-reification of the signifier into a discursive object). These could be ways of explaining the error in this scenario. However, we would not know until we have interviewed learners making the error to establish what it is that they were really thinking but these are just possible explanations. From these explanations, I can infer that the constructivist orientation is towards what is present through an orientation to the learner.

Student teachers' identification of the error

Across all 8 pairs of student teachers interviewed on this scenario, in identifying the error, they attended to learner strategies but they did this in two distinct ways. 7 out of the 8 pairs could identify the problem in the strategy and 1 pair did not. In this paper, this latter pair is not part of the discussion. The problem for the strategy was learners not distinguishing an equation from an expression, or collapsing an expression into an equation, or seeking closure. This is how the student teachers explained it in that *"the only problem that arose there was equating the expression now to 0"* (Pair 2, turn 7)[16]. All the 7 pairs pointed to this problem in some form or another.

How do student teachers then explain the equating of the expression to zero?

- The error stems from teaching emphasis

For the student teachers, the source of this problem is to a larger extent located in the teacher and to a lesser extent located in the learner. In locating the source of the problem in the teacher, there were two major explanations that were evident across all the 7 pairs. The student teachers say that the problem is with the teacher "*of not over-emphasizing certain terminologies in algebra. We have terms in algebra like simplify, terms like expression, and terms like equation, terms like solve for x or solve the equation"* (Pair 7, turn 4). From this, I infer that student teachers are saying terminologies such as simplify, expression, and solve, equation are not being emphasized enough or are not being explicit enough. In Bernstein's terms this is an orientation to what is present in the learner through an orientation to the pedagogy, hence weak framing. It is not the learner's fault here but it is that the teacher has not been introducing expressions in the right way. So the student teachers see what is present in the learner is as a result of what is absent in the teaching. That is the one explanation.

- The error stems from teaching sequences

The other explanation is the sequencing of the topics in algebra in that "*if they just started with equations, simple equation solving, that's when they go to expressions because you would think that even these expressions are equations. Maybe the sequencing should be expressions are taught first before one moves to equations"* (Pair 7, turn 6). *"You cannot start with solving equations before simplifying expressions because in solving equations there's a simplification"* (Pair 7, turn 16). From this, I infer that student teachers are saying that the source of the problem could be that the teaching of equations supersedes the teaching of expressions, and hence learners viewing expressions as equations. Related to this source of the problem is "*that some teachers think adding like terms or simplifying by adding like terms they think it's something that pupils can do they know and maybe they may skip teaching that and go straight to equations"* (Pair 7, turn 14). From this reason, I infer that student teachers are saying that the source of the problem could be that the teacher is assuming that if they teach equations learners can easily simplify given an expression. Again in Bernstein's terms this is an orientation to what is present in the learner through an orientation to pedagogy, hence weak framing. It is not the

[16] Evidence provided by an example from one pair in each case, selected for typicality across the seven interviews.

learner's fault here but it is that the sequencing has not been orderly. So the student teachers see what is present in the learners is as a result of what is absent in the curriculum.

- The error stems from learners wanting to 'solve for x' whenever they see x.

In locating the source of the problem in the learner, there was one major explanation that was evident across all the 7 pairs. The student teachers also said that the source of the problem is with the learner in that *"when there's an expression which is written in terms of x they will always want to solve for x"* (Pair 6, turn 17). *"... they will think that this problem is actually not complete because it has no equal sign. So that's why they have to put that equal to zero. That's when they will say that now this thing is complete"* (Pair 6, turn 19). From the student teachers explanation, I infer that student teachers are saying that the source of the problem is that the sight of x compels learners to solve, hence seeking closure. In Bernstein's terms this means an orientation towards mathematics – that is orientation towards absent, hence strong framing. The student teachers see something that is absent in the learners in terms of simplifying expressions.

What decisions about remediation do the student teachers suggest?

- Encourage learners to practice simplifying expressions more often

Two major strategies for remediating the error identified were evident across all the 7 pairs. One of the strategies suggested by student teachers is that *"they practice more in the simplification of algebraic expressions"* (Pair 7, turn 18). And argue that:

> *"It's not necessarily giving them 20 questions at a go, it's basically the practice. They can do 4 questions today, another day they if they are given a period in which they practice so that when they go to equations they will never confuse expressions and equations because the knowledge on simplification was consolidated"* (Pair 7, turn 20).

From this strategy suggested by student teachers, I infer that they are saying to remediate the error identified, learners should, at appropriate time intervals, practice solving of algebraic expressions. It is like the "practice makes perfect" analogy in that if learners consistently practice simplifying of algebraic expressions, the concept will be consolidated, and hence they will not confuse expressions for equations because more time has been invested in practicing. Constructivists would argue that the strategy is short term in that it addresses the error directly and not the reason for the error made. In Bernstein's terms this might be described as an orientation towards absence, hence strong framing, in that if learners do more examples they will acquire the rules for simplifying algebraic expressions. The concern here is with learner productions rather than learner thinking. And the orientation is inclined to whether the learners have got the answers right.

- Emphasizing key concepts during teaching

The other strategy suggested by student teachers pertains to explaining the meaning of key concepts. This relates to one of the sources of the error earlier discussed of the teacher not emphasizing key concepts and now they are saying the teacher has got to do so. And state that "*it's just important for you to explain the difference between an expression and the equation, so much so that they don't mix the two, because once they mix then they're wrong*" (Pair 1, turn 53). They suggest two main ways on how the distinction between an expression and an equation can be made. One way would be to "*tell them to say we are not doing this, but we just want to make these complicated expressions simpler. We are not solving them whilst finding the values for the given letters, but we just want to make them simpler*" (Pair 4, turn 36). From this suggested way I infer that student teachers are saying they would tell the learners that what they are doing by solving is wrong, and the correct way is to simplify. Alternatively, they suggest that *"it would actually help to do one where you simplify and say if we were to solve for it this is what would be there"* (Pair 4, turn 37). This is further exemplified in the excerpt below:

> *"Ya, the same one of 2x + 5 + 3x – 7 on one side. So I say, 'Let's simplify this together.' Then we simplify it. So it will come to um 5x +, I mean -, -2. Then I say the same one if I was to equate it to something then I say we solve for x. Suppose I equate it to 0, we solve for x, what do we get? Then at some point we wanna get x equal to something. Then I say, OK, on this side what we are doing is we are solving for x. On this side what we are doing is we are simplifying, and not really solving for x"* (Pair 4, turn 39).

From this I infer that the student teachers are pointing to the need to contrast expressions with what it is not, that is equations so that learners can discern the critical features of an expression. Student teachers' orientation in this case is towards what is present in the learner, hence weak framing. This is so because looking for conceptual understanding is typical of liberal progressive pedagogy – that is, a constructivist orientation to the learner. So the question here would be whether the learners have got the concepts of the difference between an expression and an equation. And the orientation here would be inclined to looking for learner explanations in terms of whether they demonstrate conceptual understanding.

So what are the student teachers' orientations?

From the scenario, student teachers recognised the error. They stated that the error is as a result of "equating of the expression to zero". The orientations of their explanations of the error identified are interesting in two ways. Firstly, they focus on learner production and what is present in this, and yet explains this as an absence, located either in the teaching or the curriculum. Secondly, to a lesser extent, the orientation is towards mathematics by focusing on absence of the mathematics in the learner. In terms of remediating learner error, student teachers' orientations are in twofold, firstly towards absence in terms of them suggesting that more practice examples are required. Secondly towards presence in terms of them pointing to the importance of conceptual understanding in the learners between an expression and an equation.

I now relate this discourse to student teachers' earlier discourse of evaluating in that it involves identifying, probing, and then correcting. So the identification of mistake is in the strategies the learners use and in the case of the scenario it is in equating the expression to zero. This identified mistake has to be corrected by the teacher because as the student teachers have stated in the scenario that it is the teacher's problem in terms of the teaching and sequencing. This points to student teachers' orientations towards presence through an orientation towards absence in the pedagogy and curriculum. So the teacher has to *"direct"* and *"rectify* by *emphasizing"* (Focus group 1). I would like to state here that student teachers overall have competing orientations towards the discourse of engaging with learner mathematical thinking in that they tend to orientate themselves towards absence and/or presence.

So the discourse of engaging with learner mathematical thinking as observed in what is legitimated by student teachers is both in the traditional and constructivist orientations. At times they tend to focus on what is absent in the mathematics so that they get the performance right and at other times they focus on what is present to push what is present towards the required understanding. What is interesting about their orientations is that they do identify error from the practical experience of the mathematics that they would know, and they talk in a language that is practical. It is very close to practice and further away from theory, which is surprising because constructivism is the thrust of their mathematics education courses. Suffice to say that theirs is an initial step towards a more comprehensive consideration. They say the error occurred because of faulty teaching. This suggests two things, firstly that there is a gap, and a lot of literature has pointed to this that theories in teacher education are not necessarily internalized by the student teachers. They can only talk in terms of those theories when they have understood them and seen the theories as personally productive.

Secondly, there is a lot of research on student errors that become useful if they can talk about errors as an overgeneralization from the constructivist perspective. Only then would they understand that re-sequencing is going to create other overgeneralizing or that emphasizing will not address the reason for the error committed. I would therefore state that student teachers' orientations for this error are in an unsophisticated manner because they do not yet have the tools to use the theory espoused in the programme. Another reason for their simplistic use of the theory may be because of their limited teaching experience. Classroom experience is necessary to concretise mathematics education theories. It is interesting that student teachers' discourses are not informed by research nor are the student teachers able to use research to help them figure out the practice. This is an indication that learner errors are not dealt with in an explicit way in this mathematics teacher education programme.

IMPLICATIONS

I would like to mention here that the implications I make here pertaining to these results are tentative in that they are a fraction of larger sets of data that are currently being analyzed. Student teachers' orientations to the discourse of engaging with learner mathematical thinking has implications for how it is dealt with in mathematics education courses offered and the attention that needs to be given, and the effect it would have on learner learning. From student teachers' orientations, it is evident that particular attention is required in the programme on how errors are thought about in mathematics and the kinds of opportunities that student teachers need to engage in order to build up their knowledge and reasoning about learner errors. If student teachers see errors as a fault in teaching either the emphasis, the order, or lack of practice exercises, or lack of contrast then it is highly likely that the error might not be corrected. This is because the source of the error is not known. This is in line with Ryan and Williams' (2007) argument that correcting the error directly without addressing the reason for the error committed does not get the error fixed. Instead, it affects the learner's intellectual mathematical development.

Where constructivism is a thrust of the programme, until student teachers become aware that errors are part of learner's conceptual understanding that is limited, and correcting them involves reconstruction on the part of the learner, then the agenda of learning is rendered futile. So the reasons for the errors need deliberate attention from either the constructive perspective and/or other perspectives that have dealt with learner error.

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REALITY BASED REASONING IN WORD PROBLEM-SOLVING

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This paper seeks to explore the frequent tendency of primary school learners to exclude real-world knowledge and realistic considerations from their solution processes when they solve real-life word problems. The study was conducted in a primary mathematics classroom context with learners drawn from different socio-cultural and linguistic backgrounds. The data collection strategies for the purpose of this study included interviews, classroom observations, and audio-visual recordings. The results of this study show that mathematics teachers should be aware of the differences between classroom mathematics and real world problem solving, and study the ways in which learners include and use cultural knowledge to arrive at and justify solutions. In brief my findings indicate that looking closely into learners' justifications of their ostensibly unrealistic solutions can inform us of the various ways in which they elucidate and make sense of the problem situation as well as the nature of problem solving activity.

INTRODUCTION

There is an overwhelmingly poor performance in mathematics word problem-solving by South African ninth grade learners in mathematics classrooms, consistent with findings of studies conducted elsewhere in the world (see for example Verschaffel, Greer, & Van Dooren, 2009). This poor peformance in word problem-solving appears to stem from the way mathematics word problems are addressed in mathematics classrooms. Learners' attempts to solve word problems reflect the mechanical methods of (or approaches to) solving word problems as promoted by the school mathematics textbooks used by mathematics teachers during teaching and assessments. In consequence, learners have a tendency to exclude reality in their solution precesses, generating conclusions that are mathematically correct but situationally inappropriate (or inaccurate) since they do not make sense in real world.

In this article therefore it is argued that it is important for all practitioners (mathematics teachers, examiners, curriculum designers, mathematics textbook authors, and learners) to be aware of the roles played by two distinct discourses in mathematics word problem-solving, namely: (1) classroom mathematics in which word problems are solved mechanically, and (2) reality based reasoning, where learners' out-of-classroom knowledge (or everyday knowledge and experience), is brought to bear on the problem-solving task. This, in turn raises the further issue of the socio-cultural (and/or linguistic) differences found in multilingual mathematics classrooms.

It is important for all practitioners to understand that the abovementioned two discourses extend across the whole domain of word problem-solving because they affect teaching, learning, and assessment in general. For example, authors and examiners should take cognisance of the type of word problems used as examples in textbooks and for assessments in examination question papers. In both cases learners' interpretations of the situation embedded within word problems may differ based on their social knowledge backgrounds, and as a result, this may advantage or disadvantage certain groups of learners.

This article begins with a discussion of the current debates on connections between classroom mathematics (or classroom activities) and second language learners' everyday experiences, and the value of connecting what is taught and learned in mathematics classrooms with the out-of-classroom learning. In particular, much of the work in this paper relies both theoretically and methodologically on notions of classroom mathematics discourse and mathematical modelling. The main argument in this paper is that implicit beliefs and rules relating specifically to learners' mathematical activities hinder learners from using realistic knowledge in their solutions. The purpose of this paper is to illustrate how learners' justifications of their ostensibly unrealistic solutions can inform practitioners, such as mathematics teachers, curriculum designers, and policy makers, of the various ways in which they clarify and make sense of the problem situation, as well as the nature of problem solving activity.

CONNECTIONS BETWEEN CLASSROOM ACTIVITIES AND REAL-LIFE EXPERIENCES

The connection between classroom mathematics and learners' everyday experiences is a complex issue because the two contexts differ significantly. Lave (1992) suggests that word problem-solving describes stylised representations of hypothetical experiences separated from the students' experiences. In word problem-solving, students' minds could be torn between two types of knowledge system that the word problem activates – one developed in the traditional mathematics classroom and the other developed through real-world experiences (Inoue, 2005). Inoue claims that in traditional schooling, students are not asked to examine different sets of assumptions for solving mathematical word problems.

For many children in elementary school, emphasis has been put on syntax and arithmetic rules rather than treating the problem statement as a description of some real-world situation to be modelled mathematically (Xin, 2009). For example, studies (Liu & Chen, 2003) conducted on 148 Chinese students from 4th and 6th grade, reported that only one fourth (26%) of the students' solutions of problems were from a realistic point of view (attending to realistic considerations). Almost half (48%) of the responses revealed a strong tendency to exclude real world knowledge, and in the rest of the cases, no answer was given. According to Inoue (2009), the ***unrealistic*** solutions may not simply stem from mindless or procedural problem solving, but could originate in students' diverse effort to make sense of the problem situation and the nature of the problem solving activity in socio-cultural contexts. In fact, Verschaffel et al. (2000) have suggested that many students whose problem solving did not seem to reflect familiar aspects of reality are known to defend their answers when their attention is drawn to the issue. Inoue (2005) argues that looking into students' justifications of their seemingly unrealistic answers can inform us of the various ways in which students interpret and make sense of the problem situation as well as the nature of problem solving activity.

Word problem-solving in school contexts serves as a game under tacitly agreed rules of interpretation (Greer, 1997). According to Gatto (1992), these agreed rules are internalised in the students' minds through the socio-mathematical norm, or hidden curriculum of traditional schooling that could influence many aspects of the intellectual activities in schools. Inoue (2009) suggests that instead of dismissing students' computational answers, examining different sets of assumptions for solving word problems can provide rich opportunities for students to learn how to use their mathematical knowledge beyond school-based problem solving. Inoue points out that this could help the students conceptualise word problem solving in terms of meaningful assumptions and conditions for modelling reality, rather than the assumptions imposed by textbooks, teachers, or authority figures.

THEORETICAL PERSPECTIVE

From a socio-cultural perspective, modelling implies engaging in inter-semiotic work. In other words, one has to decide about the appropriate and useful manners of coordinating linguistic categories and mathematical expressions and operations in order to come to a solution problem (Säljö, Riesbeck, & Wyndham, 2009). In inter-semiotic meaning-making, the truth value of statements and arguments is established on the basis of analytical considerations of how a particular usage of concepts fits into the universe of meaning that is mathematical discourse.

METHOD

Participants

Participants in this study were 40 ninth grade learners from a senior secondary classroom in Port Elizabeth. These learners were drawn from different divergent social-cultural backgrounds. The school attracts learners from poor to low-income households, and families receiving social grants from the government.

Data gathering included tests and video recordings, supported by field notes. Learners' responses during group and individual problem-solving of the problem-solving task were video-taped and later transcribed in full. These methods of data gathering are appropriate for research designed within a socio-cultural perspective because they allow for the opportunity to examine classroom discourse.

Materials

In this study, the learners were given a word problem-solving (PS) task, whose solutions depended on realistic factors connected to the problem situation. The PS task was the modelling problem adapted from Verschaffel et al. (2009). The researcher was present throughout the problem-solving process, and learners were encouraged through questioning to verbalise and/or write down the reasoning process they employed to arrive at and justify a particular solution.

Problem solving (ps) tasks

PS1: Two boys, Sibusiso and Vukile, are going to help Sonwabo rake leaves on his plot of land. The plot is 1200 square meters. Sibusiso rakes 700 square meters during four hours and Vukile does 500 square meters during two hours. They get 180 rands (R) for their work. How are the boys going to divide the money so that it is fair?

PS2: John’s best time to run 100 meters is 17 seconds. How long will it take him to run 1 kilometre?

Procedure

The two problem-solving (PS) tasks were read aloud to all groups of learners by the researcher. All the learners also received the task in written form. After attempting the problems individually, learners were then introduced to discussion as a strategy to make sense of word problems, and engage fully in classroom discourse. The group interactions were videotaped and later analysed. The researcher was readily available throughout the session in order to neutrally assist in cases where learners were stuck and to record verbal reasoning from different groups, and learners’ justifications of their responses.

RESULTS

The word problems above, which are examples of a central part of mathematics learning, can be seen as attempts to connect mathematical reasoning to everyday life. In other words, the PS task can be viewed as a manifestation of the notion that mathematics is or should be part of mundane practices in everyday life. The results of this study illustrate that students acted in a complex situation when attempting to solve these problems in an ambiguous reality. Learners responded to PS1 and PS2 tasks by using different models and approaches (see Extracts below) that illustrated different interpretations and use of real world experiences in their problem-solving.

Results of PS1

The following extract shows how learners interpreted and solved the first problem (PS1), and describes their reactions after being prompted by the researcher.

<u>Extract 1.1</u>

Learner 5: Because they both worked, I will just give them the same amount.

Researcher: Any other suggestion? How do you think they should divide the money fairly amongst themselves?

L5: I will still share it equally because there's no need to take more money than her.

R: OK...

L6: To be fair I will give the one who raked 700 meters R100 and the other one R80.

R: What do you think about he who raked for 2 hours?

L1: It is not fair,... one did it in shorter time ..., and the one worked in 4 hours and did 700 square meters, so will first have to calculate the time and then divide up.

In general it was observed that all participants found the problem very difficult to solve. Learners encountered difficulties in making sound and reasonable assumptions concerning what it means to share or divide the money in a fair manner. The discussions are characterised by learners engaging in realistic considerations (Verschaffel et al., 2000), as a result of solving a problem in an ambiguous reality. From some of these learners' socio-cultural perspectives, dividing the money fairly simply translates to sharing the money equally, as seen in the text of extract 1.1, where learner 5 said: "*Because they both worked, I will just give them the same amount*". The fact that this statement could legitimately be challenged by other learners suggests that learners enter mathematics classrooms from a range of socio-cultural backgrounds. Interestingly, dissatisfaction with this response also suggests that there is a specific culture of calculation represented in and through the practices embedded within the mathematics classroom, and learners whose socio-cultural background is congruous with that classroom culture are more likely to be constructed as successful learners (Zevenbergen, 2000).

Mathematising as communicative work

The data show that learner discussions during the interviews moved back and forth between the problem-solving strategies that they employed. What is evident from the texts in extract 1.1 is that culture and real-life knowledge played a pivotal role in learners' mathematical reasoning and problem-solving in relation to this task. The table below, Table 1.1, shows different models that were used by the learners in different groups, when attempting to solve this task.

Table 1.1: Models suggested for sharing money in learners' responses

Models for sharing	No. of learners (n=40)
A. Divide equally (R180/2)	26
B. Amount of work done	12
C. Time taken to do work	2
D. Payment by performance	0

The data in the table above reflect a highly frequent response in which learners propose sharing the money equally. One may legitimately assume that, as noted earlier, the high number of responses suggesting that the best solution is sharing the money equally stems from a specific interpretation (out of multiple meanings) of the word ‘fair’, influenced by learners’ socio-cultural backgrounds.

Calculations using magnitude of work done

It is very interesting to see that there were further suggestions, beyond sharing or dividing the money equally, which can be taken as an indication that learners used diverse classroom mathematical experiences and real-knowledge skills acquired through life-experiences. The data show that only boys suggested the alternative model of sharing the money by calculating the amount of work done (without considering the time taken as a factor). Although one cannot make conclusive claims from the data, it is well-known that in African cultures, there are certain jobs that are only reserved for boys. Practical experience of the labour involved might explain the higher demand on the part of boys for a different kind of distribution. In short, boys and girls probably used different real-life knowledge and experiences in suggesting a model to solve this problem.

<u>Extract 1.2</u>

L6: To be fair I will give the one who raked 700 meters R100 and the other one R80.

L7: If we divide the piece of work done by the total ground that was raked, we have 700 divide by 1200, which gives 7/12. Then we multiply 7/12 by R180, the money to be shared, which gives R105. So one should have R105 and the other one gets R75.

The text in extract 1.2 shows how two boys solved this problem. Both learners 6 and 7 considered the amount of work as a key factor of sharing the money fairly. It is clear that Learner 6 estimated the proportions of the money to be shared. His estimation is not far from learner 7’s solution statement. Learner 7 used the concept of decimal fractions to solve this problem based on amount of area raked by each boy, and sharing the money according to the fraction equivalent to the work done. It is also evident that language had no effects in learners’ interpretation of this problem statement.

Table 1.2: Comparison of PS results per country and per target group in this study

Models for sharing	**No. of learners**	
	Sweden *n=78*	*Target study* (n=40)
A. Divide equally (R180/2)	33 (42%)	26 (65%)
B. Amount of work done	27 (35%)	12 (30%)
C. Time taken to do work	18 (23%)	2 (5%)
D. Payment by performance	0 (0%)	0 (0%)

Compared with the situation in which Western students (e.g., Sweden and United States) have been challenged by the problematic word problems (Greer, 1997; Säljö te al., 2009; Verschaffel et al., 2009), the data in Table 1.2 above show that some South African learners are not in the position to, as Freudenthal (1973) puts it, "mathematise" the world by means of elementary forms of mathematical modelling. The data in the table above shows that the majority (65%) of the South African learners in this study failed to argue and make counter-arguments beyond equating a "fair sharing" of the money with "dividing money equally". Relatively few (35%) of these learners engaged in further modelling approaches, compared with 58% of the Swedish students, who reportedly employed multiple modelling approaches to solving this PS task.

The following extract reflects the recorded arguments made by the ninth grade learners when engaging in solving the PS1 task. The task was given to the learners after they were introduced to discussion and argumentation as a strategy to engage in problem solving, and connecting mathematics classroom with the outside world.

Extract 1.3: *Learners' group interactions when solving a PS task*

Model			
Amount of work done	**Time**	**Pay by performance**	**Other**
L(learner)4: They both worked, I'll just give them same equal amount.			
	L1: …the other one did in shorter time…so we'll have to calculate the time and then divide up.		
L6: To be fair, I'll give the one who 700 square meters R100 and the other one R80			
	L5: ...one worked the smallest part in a short time, so the other one used much more time working in a bigger place...		
		L5: ...Vukile must have more than R90 because he did it in a very short space of time...	
L2: No...Vukile only did less work...			
		L5: I agree, but he did it faster than Sibusiso...	
			L3: Because they are friends, I will share it equally because there is no need to take more

money....

L2: But Sibusiso raked for four hours and Vukile for just two hours, how can you share money equally?

L4: R180/2 is R90 for each of them, it's fair...

The text in Extract 1.3 is an example of one of the episodes recorded during PS group discussions and interactions. During this activity, learners were encouraged to discuss in the language of their choice, and they used predominantly English to solve the task, with rare occasions of code-switching. Data in this extract illustrate that learners moved between the models and computations without noticing what premise applies in each case. The utterances in this extract clearly suggest that learners' reasoning occurred between the proposed models, with each member of the group failing to reason beyond one model. As such, the counterarguments against one model often came from a different model without comprehending that the premises for the reasoning and calculations have changed.

In the fourth and fifth utterances in extract 1.3, we see how L5 moves between the two models without taking into account that they are different in terms of their propositions and implications to fair sharing of the money between the boys. Although all the group members (six learners in this group) participated in dialogue and talk, they could not arrive at a common solution to this problem. Rather, it was evident that the quality of arguments and nature of justifications improved over time, as they continued to engage in mathematical modelling of this problem.

Results of PS2

Word problems are often the only means of providing learners with basic pragmatic or common-sense experience in problem-solving and mathematization (Reusser & Stebker, 1997). The PS2 in represents one of many questions that are used for assessments in South African mathematics classrooms, and elsewhere in the world (see Verschaffel et al., 2009). All the learners' solutions were classified into three main categories based on their written answers and verbal responses to the interview questions. In fact, the PS2 task has a mathematical structure that is related to real-life factors. In other words, the solution of this problem depends on the rate of progress influenced by factors such as physical strength, preparedness, weather, fatigue, etc.

In solving the PS2 task, learners failed to reflect common-sense understanding of reality in problem-solving. The majority of learners answered *"170 seconds"* to this problem, consistent with findings of many studies conducted in Europe and Asia, for a wide variety of problems across different linguistic and cultural settings (see for an example Schoenfeld, 1991; Verschaffel et al., 1994, 2000, 2009). In this case, learners simply read and converted the text into a mathematical operation in fairly direct manner, without considering more carefully in what manner the text information is to be translated into a mathematical form in order to be successful. In responding to this problem, learners simply multiplied 17 seconds by 10 to find out how long it takes to run a kilometre. This "runner problem" has been used in number of studies in different parts of the world, and the results, consistent with the results of this study, are thought-provoking: "...the percentage of students in the various countries who gave the unqualified answer 170 seconds ranged from 93% to 100%" (Verschaffel et al., 2000, p. 44).

The nature of justifications

The extract below demonstrates learners' responses to PS2 activity, and the nature of justifications made by learners before and after being prompted by the follow-up interviews.

Extract 2.1

L5: I multiplied 17 by 10 it gives me 170, and then I got my answer.

R: Your solution may not work, because of real life factors. Why did you solve the problem that way?

L1: I don't agree with **L5**, because it's a kilometre, when you run 100 meters you run with your full speed, but then at 17 you cannot run your full speed, you have to a bit sometimes jog because this is a kilometre it's not 100 meters...

R: Okay...

L1: Mathematically it's correct but in real life it's not going to be like that, it's going to be much longer, it's not going to be a 170 seconds.

The learners' responses to the researcher's question in extract 2.1 demonstrated that learners could justify their responses in terms of their own interpretations of the problem situation when confronted with, what Inoue (2009) refers to as, the "irrationality of their responses". Extract 2.1 shows the justification that learner 1 presented after she was prompted to do so in the interview question. Learner 1 suggests that although the solution is mathematically correct, "*in real life it's not going to be like that*", as she reasons that in a real life situation it will take John "*much longer*" to run the 1 kilometre distance.

This learner acknowledges the disconnect that exists between what she learnt in classroom mathematics and real-life problems that are not related to the mathematics discourse that she is exposed to. In so doing, her newly acquired argumentation and discussion skills assist her in affirming the mathematical solution offered by learner

5, and in the process linking the mathematics to real knowledge by suggesting that "*...when you run 100 meters you run with your full speed, but then in this case you cannot run your full speed, you have to jog a bit sometimes, because this is a kilometre it's not 100 meters*... This reasoning is largely influenced by consideration of realistic factors that exist in real life situations. As can be seen in Table 4.7 below, there were different justifications of seemingly 'unrealistic' responses. These responses reflect a sample of justifications that were presented by the learners spontaneously (in response to the second interview question) as well as examples of justifications after being prompted explicitly.

Table 1.3 Learners' sample justifications of 'unrealistic' responses

Spontaneous justifications	**Justifications after being prompted explicitly**
John is a well trained runner. He can make this time.	I don't think that is possible, because if run too much you will get tired and your speed will decrease, so as the speed slows down the time goes bigger.
If he is fit as expected, John can maintain his best 100m time in a kilometre distance.	When you run 100 meters you run with your full speed, but then at 17 you cannot run your full speed, you have to a bit sometimes jog because this is a kilometre it's not 100 meters
	It's not true because in the first 100 meters you running your full speed, but when the time goes on you get tired
	If John is a super-fit athlete, who trains regularly with a coach, then he can make it on time

Similar to studies conducted by Inoue (2009) on an introductory-level psychology class in Southern California, most of the learners' justifications were based on the claim that common-sense real life factors do not necessarily apply to certain or particular mathematical and/or classroom situations. In so doing, justification of computational answers were designed to make their responses reasonable and acceptable.

The data showed that these learners are used to this kind of problem, particularly in natural science studies. Moreover, the text in extract 2.1shows that learners are exposed to classroom settings where simply providing the "correct" answer to a structured problem is sufficient. This was noticed when learner 5 answered: "*I multiplied 17 by 10 it gives me 170, and then I got my answer*", without providing justifications and checking whether the answer is reasonable. A prominent finding in most of the research of this kind is that learners' performance on word problems differs dramatically depending on how the problems are designed (see Verschaffel et al., 2000). The PS2 task is formulated according to the standard expectations in mathematics teaching, and, within this discourse, it can be solved correctly through a straightforward operation such as division or multiplication.

DISCUSSION

It is fundamentally necessary to draw reality into mathematics classrooms by starting from learners' everyday-life experiences and situations, if one aims to teach learners to connect classroom mathematics to real-life knowledge in their thinking and reasoning. The inclusion of application and modelling problems is intended to convince learners to develop the necessary skills of knowing when and how to apply their classroom mathematics effectively in situations encountered in everyday life. I contend that this goal can only be realised if learners and teachers bring reality into mathematics (that is, view everyday life situations and learners' experiential reality as a natural extension of teaching and learning formal mathematics) and conversely bring mathematics into reality. I believe that engaging learners in mathematical reasoning, using available and relevant real-world contexts that are familiar to them and/or related to their own daily experiences permits them to deepen and broaden their understanding of the usefulness of mathematics, and may influence sound mathematical conclusions that make sense in out-of-classroom contexts. In other words, learners should be regularly encouraged to identify an immense variety of situations as mathematical situations, in the process learning a variety of ways of thinking mathematically.

Exposing learners to word problems that are familiar to them, like the problems used in this study, may be viewed as an attempt to establish a new classroom culture through new socio-mathematical norms. Such problems provide learners with the opportunity to model and 'mathematise' a problem situation, and not primarily to apply a ready-made solution procedure without realistic considerations. This is not at all to implythat knowledge of solution procedures is not relevant. It serves only to stress that the primary objective is to make sense of the problem. As noted earlier in this paper, learners were encouraged to use and justify their own sense-making methods, exploring the usefulness and soundness of their suggested models with regard to the problem. In the process of presenting arguments and counterarguments, learners are stimulated to articulate and reflect on their cultural or personal beliefs, alternative conceptions and effective strategies to solve problems.

CONCLUSION

In this paper, I have discussed an example (word problem-solving) of the kind of classroom activity that connects classroom mathematics to everyday-life experiences and knowledge of the learners. I view the introduction of new socio-mathematical norms in mathematics classroom as an attempt to create a substantially reflective teaching and learning environment. Teachers should be aware of the fact that the context of mathematics schooling and the real world context are fundamentally different. What this study tried to illustrate is that mathematics word problems become more complex when the relationship between the mathematical operations and the verbal formulations are not of the standard kind (Säljö et al., 2009).

Mathematics classrooms in South Africa consist of learners from different cultural and social classes. Individuals in these classrooms are engaged in different kinds of discourses that sometimes overlap and at times are mutually exclusive. Consequently, classroom stakeholders are faced with challenges of choosing discursive practices that promotes mathematical problem-solving and arguing in a particular setting. Overlaps and/or moves between discourses in the mathematics classroom is sometimes complicated, as illustrated in this study in relation to the concept of 'fair sharing of money', discussed earlier in this paper. In fact, learners find it very difficult to assess what the logic of the argumentation is and what are useful arguments at a particular point in time.

The pedagogical conclusion of this study is that cognitive power produced by multilingual mathematics classrooms settings has a strong influence on learners' realistic problem solving, and implicit beliefs and rules relating specifically to learners' mathematical activities, hinder learners from using realistic knowledge in their solutions. Finally, this study illustrated that looking closely into learners' justifications of their ostensibly unrealistic solutions can inform us of the various ways in which they elucidate and make sense of the problem situation as well as the nature of problem solving activity.

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AN EXAMINATION OF PUPILS' PERFORMANCES ON COMPUTATIONAL-TYPE AND PROOF-TYPE GEOMETRY PROBLEMS - A PILOT STUDY

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The paper is located within the problematic of the constitution of mathematics in the pedagogic situations of schooling, and derives from work done to prepare thirty Grade 12 pupils, from seven schools (3 former-DET, 1 former-CED and 2 former-HoR high schools) in the Western Cape for the 2010 Mathematics Paper 3 matriculation examination. For this paper we are concerned with and report on the differences in performance of pupils on Euclidean geometry problems that ask pupils to calculate some or other quantity, and those problems asking them to prove some or other proposition. We describe the type of resources pupils draw on to regulate their mathematical activity as they deal with the two types of problem. In the main, we find that pupils exhibit competence in problems demanding the calculation of quantities, but that they struggle to develop deductive proofs, even though they, apparently, regulate their calculations by drawing on the same propositions that they would need to use to construct proofs. The paper sets out to construct a research problem focusing on the differences in performance of the two problem types.

INTRODUCTION

In this paper we consider aspects of the performances of thirty Grade 12 pupils from seven high schools in greater Cape Town who enrolled in a programme to prepare them for the 2010 Mathematics Paper 3 matriculation examination. The content examined in Paper 3 was not offered at their schools. The teaching sessions suggested that: (a) some pupils had a very limited, if any, exposure to Euclidean geometry; (b) pupils' mathematical activity was generally not regulated by explicit appeals to propositions; and (c) pupils had not being taught to read and write proofs.[17]

It is envisaged that an analysis of pupils' general success in solving problems demanding calculations and their difficulty in answering problems requiring the production of proofs, will enable a more nuanced understanding of their contrasting achievements. The ultimate purpose of this paper is to construct a researchable problem with respect to pupil's responses to two different types of geometry problems.

[17] Nevertheless, 22 of the 30 pupils did get to pass the final 2010 Grade 12 Mathematics Paper 3 examination. The pass rate could be attributable to pupil performance in Probability and Data Handling. Verification is not possible since we do not have access to the final examination scripts.

The terms *computational-type problem* and *proof-type problem* are used to distinguish between the two types of problems that pupils encounter in their engagements with Euclidean Geometry problems at high school. Computational-type problems refer to those questions that explicitly ask pupils to calculate the sizes of angles or lengths of sides or areas of geometrical figures, given quantitative data on lengths or/and angles or/and area. Angles can be calculated in degrees or in terms of a given variable; the lengths of sides can be calculated in specific units of length or some or other variable; and area can be calculated numerically or in terms of a variable. In solutions to proof-type problems pupils are expected to generate arguments consisting of strings of propositions that establish the mathematically necessary intermediate and final results that constitute a proof as a whole. In such problems relationships between angles and sides of geometrical figures are given, instead of the actual sizes of angles and lengths of sides.[18] While we distinguish between the two problem types, it should be remembered that the calculations in all *computational-type* problems are, ultimately, regulated by propositions. For example, when calculating the size of an unknown angle in a triangle, a pupil may have to use the proposition: *the sum of the angles of a triangle is equal to 180º* - or any other suitable propositions(s).

In computational-type problems, the sizes of angles and lengths of line segments are given degrees and length units, respectively, accompanied by information about other geometrical objects (lines and angles) - much of the information is presented in graphical form rather than in language. The values of various angles have to be calculated by using the information given, subject to the relevant propositions indexed by the drawing and explicitly stated in language. In proof-type problems the magnitudes and lengths of angles and sides, respectively, are not given. Once again, a great deal of information is provided graphically rather than in language. Pupils are expected to engage in more formal forms of proof to solve the problem. In both problem types pupils are, of course, expected to provide propositional support for their calculations and arguments.

Before we elaborate on the research methodology used to explore the phenomenon described, we briefly discuss a selection of ideas from the literature that we find productive for thinking about the issues that pertain in this instance.

VAN HIELE AND PROPORTIONAL REASONING

To analyse and describe actual pupil answers, Van Hiele's (1986) theory of geometrical thought and the cognitive construct of *proportional reasoning* are employed as analytic resources.

The van Hiele theory provides us with a framework to analyse pupils' geometrical thought. Van Hiele postulated that children sequentially progress through the levels

[18] Copies of the actual examination question paper will be available to attendees at the conference.

of geometrical thought, and named these levels *visualisation*, *descriptive*, *informal deduction*, *deduction* and *rigour*. At the first level, visualisation, the child identifies shapes based on appearance – thinking is nonverbal and properties of the figures are not recognised. At the next level, descriptive, the properties of some objects are identified and a linguistic orientation, which is more abstract, is necessary to give a description of the shape. At the level of informal deduction, the pupil is able to logically argue about the properties of shapes or the relations among the properties, that is, definitions are given in terms of minimum sets of properties and discovers new properties by deduction. For example, a figure may be described by an exhaustive list of properties at the descriptive level. At the informal deduction level it is possible to select one or two properties of the figure to determine whether these are sufficient to define the figure. According to van Hiele, if the structures of the visual, analysis and informal deduction are not clearly grasped, the pupil will not learn from the traditional deductive approach expected at the fourth level of his hierarchy of reasoning. At this level, a pupil recognises and flexibly uses the components of an axiomatic system, but cannot compare axiomatic systems. Olkun *et al* (2005) asserts that when school geometry is presented axiomatically, false assumptions are made about the ability of pupils to operate in a formal deductive manner, since fundamental understandings of geometry may be lacking. The final level is not attained at school and is located in the domain of tertiary studies in geometry.

A second analytic resource, namely, *proportional reasoning*, will be used in the discussion of the data, with particular reference to the problems in Question 1 of the examination paper under review. According to Lamon (2007), education researchers tend to describe proportional reasoning in terms of comparison problems and missing-value problems. Both types of problems involved an order relation between the ratios of four quantities a, b, c and d, i.e., $\frac{a}{b}\ (< \ = \ >)\ \frac{c}{d}$. See Lamon (2007) for examples of comparison and missing value problems. In arguing that proportional reasoning cannot merely be defined in terms of an understanding of the structural relations in comparison and missing-value problems, she proposes that "*proportional reasoning* means supplying reasons in support of claims made about the structural relationships among four quantities, in a context simultaneously involving covariance of quantities and invariance of ratios or products" (Lamon, 2007: pp. 637, italics in original). This would include an awareness of the multiplicative relationship between two quantities and the ability to extend the relationship to two other quantities (pp. 637 – 638). But Lamon also cautions that an ability to solve a problem is no guarantee that proportional reasoning is indeed taking place. The correct answer may precipitate from a procedural manipulation or from the application of knowledge about equivalent fractions.

METHODOLOGY

Having recognised that the pupils appeared to be more successful when producing solutions to computational-type problems, but performed rather poorly on problems demanding proof arguments, we decided to use the mid-year geometry examination[19] to explore the apparent phenomenon. First, we needed to check that there really was substance to our sense that the pupils perform differently on different problem types, even though the same propositions and definitions support both types of problem. Second, we wanted to examine in some detail how students treat the two problem types, specifically to check whether there is some general manner in which they attempt to solve problems that assists them in producing solutions to computational-type problems, but which disrupts the production of solutions to proof-type problems. Third, we wanted to generate ideas for how to proceed with further research on the phenomenon, preferably with a larger sample of pupils and across different types of school.

With our three purposes in mind, the mid-year examination paper was constructed so that (1) it contained both computational- and proof-type problems (sometimes both types occur within a single, multi-part problem), and so that (2) it contained

Propositions	Computational-type	Proof-type
The line parallel to one side of a triangle divides the other two sides proportionally	3	2
The corresponding sides of two similar triangles are in proportion	1	2
The angle between a tangent and a chord is equal to the inscribed angle subtended by the chord	1	3
The base angles of an isosceles triangle are equal	2	1
The sum of the angles of a triangle is equal to 180º	2	1
The opposite angles of a cyclic quadrilateral are supplementary	1	1

Table 1: Number of computational- and proof-type problems matched in terms of expected proposition usage

computational- and proof-type problems requiring the use of the same propositions and definitions. Table 1 lists the number of problems sorted into computational- and proof-types, showing those problems across the two categories that can be thought of as requiring the use of the same propositions and definitions. To enable a discussion with regard to pupils' performance in the two types of problems, certain questions were selected as examples of computational-type and proof-type problems. Table 1 shows that pupils would have had to draw on similar propositions when solving the problems. Examples of propositions in circulation in both types of problems are *the opposite angles of a cyclic quadrilateral are supplementary*, *the corresponding sides*

[19] Further studies will have to draw on a more extensive data archive – see Conclusion.

of two similar triangles are in proportion and *the angle between a tangent and a chord is equal to the inscribed angle subtended by the chord*.

The examination scripts of the thirty Grade 12 pupils were then analysed in various ways to generate data.

A question-by-question analysis of marks scored by each pupil

First, a measure of each pupil's performance on each of the examination problems was recorded in terms of the marks they obtained for each problem, so also giving us a measure of their problem-specific, as well as their overall, performances in terms of problem-type. The results of that analysis are displayed in Table 2.

The data indicates that 21 of the 30 pupils (70%) were able to attain a mark in of 50% or more for the computational-type questions.

However, some pupils scored very few marks on computational-type problems in general, or on specific problems falling in the category. Seven pupils failed to attain 30% or more on the computational-type problems. The mean percentage for the

	Problems									
	Computational-type					Proof-type				
	1.1	1.3	2.2	2.3.1	Σ	1.2	2.3.2 2.3.3	3.1	3.2	Σ
Max. poss. score	8	8	8	4	28	6	9	15	11	41
Mean score	4,7	3,4	6,2	2,4	16,8	3,2	3,7	3,8	1,9	12,6
Mean %	58,8	42,5	77,5	60,0	60,0	53,3	41,1	25,3	17,3	30,7

Table 2: Summary of marks scored by each pupil across the categories of problem

individual computational-type problems ranges from 43% to 78%, with a category mean of 60%.

Low achievement scores were recorded for pupils on the proof-type problems, while a few even scored zero. Only eight pupils (approximately 27%) passed at the 50% mark. An analysis of the actual answers in the scripts shows that many pupils did not attempt, or could not answer, the proof-type questions. It is, of course, possible that those pupils may not have had sufficient time to answer these questions if they devoted too much time to the computational-type problems which appeared earlier in the question paper.

In contrast to the computational-type problem statistics, the mean percentage of pupil achievements on proof-type problems ranges from 17% to 53%, with a category mean of 31%. In this case, the standard deviation of 9,8 places 20 pupils within one standard deviation of the mean. From the statistics it emerges that the ratio of the mean percentage for the computational-type problems to the proof-type problems is almost 2:1. But the ratio of percentage passes in the computational-type to proof-type problems is almost 3:1 (70:27). Therefore, for this group of pupils, almost three times as many pupils show greater competence in computational-type problems than in proof-type problems. There were eight instances of pupils performing poorly on both types of problems and one instance, pupil (L10), of better achievement on the proof-type problems over the computational-type problems.

This concurs with our initial observations during teaching sessions and raises the question: What resources do pupils use to solve these different problem types? This is the focus of the next section. So, to facilitate further descriptions of pupil achievement across the two types of problems, some actual examination answers are now discussed in an attempt to begin to understand why, in the main, a greater number of pupils were significantly more successful in answering computational-type problems than proof-type ones.

Extracts from examination scripts

We now consider actual pupil answers to the questions relating to Ratio and Proportion in Triangles and Circle Geometry questions, respectively.

Q1.1 and Q1.3 were classified as computational-type and Q1.2 as proof-type.

In Figure 1, the pupil draws on propositional statements to regulate the statements of proportionality even though the proportion statements in 1.1.2 and 1.3 are incorrect and successfully completes Q1.1.1, a computational-type problem, and Q1.2, a proof-type problem. Solving for *x* by inspection in Q1.1.1 may indicate that the pupil is using the construct of proportional reasoning as a resource to solve the problem.

1.1.1 $\frac{AE}{AB}=\frac{CD}{BC}$ ….. ED || AC $\frac{x}{4+x}=\frac{3}{x+1+3}$ $\therefore x=3$	1.1.2 $\frac{ED}{AC}=\frac{AE}{EB}=\frac{CD}{BC}$ ….. ED || AC $=\frac{3}{7}=\frac{3}{7}$ $\therefore \frac{ED}{AC}=\frac{3}{7}$
1.2 In ΔPRT $\frac{PW}{PT}=\frac{PQ}{PR}$ ……. QW || RT In ΔPSR $\frac{PT}{PS}=\frac{PQ}{PR}$ ……. QT || RS $\therefore \frac{PW}{PT}=\frac{PT}{PS}$	1.3 $\frac{AF}{GF}=\frac{AC}{AB}=\frac{CF}{EF}$ $\frac{AF}{GF}=\frac{CF}{EF}$ ……. AF || BE $\frac{5m}{2m}=\frac{30}{EF}$ $\frac{5}{2}=\frac{30}{EF}$ EF = 12 units

Figure 1: Extracts of answers to Q1 from the script of L29

The pupil's successful completion of the computational-type problems is contradicted in the way that he/she is not able to string together a sequence of coherent statements in Q1.1.2 and Q1.3. This raises the question: Is the pupil really drawing on an axiomatic system to validate a sequence of logical, coherent statements based on deductive thought, or is something else happening here? A second set of actual pupil answers is analysed below.

The responses to Q1 (in Figure 2) by pupil L12 show that generally he/she does not provide validation statements and does not recognise the need to validate answers. The answer to Q1.1.1, $x = -4$ is not declared invalid. The pupil has not related the solution to the problem, that is, *x* represents the length of a line segment and therefore can't be negative. The procedure seems to regulate the solution rather than the line segment as an object. Further, the procedure clearly indicates that the pupil is reasoning but using a mechanical process of cross-multiplication. He/she may have learnt that 'when you have two fractions equal to each other, you cross-multiply'. As in the case of L29 (Figure 1), and in other instances of data in the archive in respect of Q1.3, this pupil shows evidence of lacking the necessary cognitive resources (like proportional reasoning) to develop proportional relationships, from which a logical, coherent geometrical argument can be derived. What is presented appears merely to be a consequence of the recognition of a set of structures that iconically generate a particular relationship. That is, because a line segment is parallel to a side of the triangle, there exists a proportional relationship between the lengths of the line segments constituting the other two sides of the triangle. However, after this is recorded, there is no further evidence of deductive reasoning to produce a logical, coherent sequence of statements.

Some circle geometry questions are now analysed and described, and we draw on the levels of geometrical reasoning (Van Hiele, 1986) as analytic resources for these descriptions.

1.1.1 $\frac{BD}{DC}=\frac{BE}{EA}$ $\frac{x+1}{3}=\frac{4}{x}$ $x^2+x=12$ $x^2+x-12=0$ $(x-3)(x+4)=0$ $x=+3$ or $x=-4$ 1.1.2 $\frac{ED}{AC}=\frac{AE}{EB}=\frac{CD}{DB}$	1.2 $\frac{PW}{PT}=\frac{PQ}{PR}$ and $\frac{PT}{PS}=\frac{PQ}{PR}$ $\therefore \frac{PW}{PT}=\frac{PT}{PS}$	1.3.1 FE is 2 (because $GE \parallel AS$) $\frac{FE}{ED}=\frac{FG}{GA}$ 1.3.2 $\frac{AB}{BC}=\frac{FE}{EC}$ (because $AF \parallel BE$) $\frac{AB}{BC}=\frac{2x}{27x}$

Figure 2: Extracts of answers to Question 1 from the script of L12

In Figure 3, the pupil strings together an appropriate sequence of statements in the calculations to generate correct angle values and draws on propositions to validate statements, although deficiencies exist in the precise wording of some of the propositional statement. This pupil has achieved the van Hiele level of ***informal deduction***, evidenced in the way that properties of geometrical figures and the relations among these properties are strung together. However, the pupil is largely unsuccessful in answering Q3.1, with the exception of Q3.1.2. The successful completion of Q3.1.2 could have emanated from the recognition of an iconic resource (similar teaching examples) generated during the teaching sessions. That is, an imagistic feature could possibly have triggered a solution strategy. From worked examples presented in class, pupils become familiar with the solution format of problems involving similarity in triangles. Predictably a proportion in respect of sides of triangles will precipitate from a question requiring pupils to show that two triangles are similar. Given that the pupil was unable to answer Q3.1.4, which involves deducing the proportion from two similar triangles, it is highly likely that the student derived the proportion $\frac{AB}{AD}=\frac{BC}{DB}=\frac{AC}{AB}$ directly from the Q3.1.1 statement by using notational convention rather than from the corollary on proportional sides that follows similarity of two triangles. Consequently, it is not surprising that the pupil could not produce a proof for Q3.1.4, in which he/she is expected to prove that

2.2.1 $\hat{D}_1 = \hat{B}_1 = 40°$ .... exterior ∠s = the opp inter ∠ $\hat{B}_1 = \hat{D}_2 = 40°$ .... base ∠s of an isos are = $\therefore B\hat{D}C = 40°$ 2.2.2 $\hat{C} + \hat{B}_1 + \hat{D}_2 = 180$ .... sum of ∠ = 180 $\hat{C} + 40° + 40° = 180$ $\hat{C} = 100$ $\hat{C} + \hat{A} = 180$ .... inter opp are = 180° $100 + \hat{A} = 180$ $\hat{A} = 80°$ 2.2.3 $\hat{O}_1 = 2\hat{A}$ $= 2(80°)$ .... the centre theorem $\hat{O}_1 = 160°$	3.1.1 ΔABC and ΔADB $\hat{A} = \hat{A}$ .....common $\hat{C} = \hat{B}$ .... given $\hat{B} = \hat{D}$ $\Delta ABC \parallel\!\mid \Delta ADB$ .... ∠∠∠ 3.1.2 $\Delta ABC \parallel\!\mid \Delta ADB$ $\frac{AB}{AD}=\frac{BC}{DB}=\frac{AC}{AB}$ ..... corollaries $AB^2 = AD.AC$ 3.1.3 ΔACE and ΔAED $\hat{A} = \hat{A}$ .....common $\hat{C} = \hat{E}$ .... given $\hat{E} = \hat{D}$ $\Delta ACE \parallel\!\mid \Delta AED$ .... ∠∠∠ 3.1.4 AB = AE

Figure 3: Extracts from the mid-year Grade 12 Paper 3 examination script of L18

$AE^2 = AC.AD$ using the similarity proven in the previous question, before deducing the equality of two line segments (AB = AE).

The attempts at answers to Q3.1.1 and Q3.1.3 suggest that problems requiring formal deductive reasoning and proof are a challenge to this pupil. The notations offered for Q3.1.1 and Q3.1.3 indicate that he/she knows that pairs of angles must be equal for the triangles to be similar; also, that the corresponding pairs of angles must be equal. Schooling also tends to insist that the triangles are named in a particular way so that the production of the relationship of the sides is automatic. Consequently, the pairs of angles appear to have been extracted from the statement of similarity. This could be an indicator that the pupil does not understand the significance of deduction and the role of axiomatic systems and, consequently, we could conclude that he/she has not achieved van Hiele's fourth level of ***deduction***.

In Figure 4 this pupil is able to reason deductively and present a logical, coherent argument, in the way the statements are presented in the answers to Q2.2, and draws on propositions to validate computational statements in both Q2.2 and Q3.1. In other words, referring to the actual objects and the related propositional statements regulates the mathematical activity. The pupil's answers to Q1.1 and 1.3 (not shown here) further indicate a competence in computational-type problems. Notwithstanding the successful completion of Q3.1.1 and Q3.1.2, there is evidence of a breakdown in the general ability to successfully complete proof-type problems. Table 2 indicates that this pupil (L05) did not score any marks in Q2.3.2, Q2.3.3 and Q3.2. Strangely, this pupil, after successfully completing a proof for Q3.1.1, appears to use 'congruency' strategies to similarity. Once again, it is possible to conclude that formal deductive reasoning and proof has not been mastered, and deduction and the role of axiomatic systems are not fully understood.

2.2.1 $\angle D_1 + D_2 + D_3 = 90°$ $\hat{D}_1 = 40°$ ..... given $\therefore D\hat{B}C = 40°$ ..... tan chord theorem $\therefore B\hat{D}C = 40°$ ..... base $\angle$s of isos Δ	3.1.1 $B\hat{A}C = B\hat{A}D$ ..... common $C\hat{B}A = B\hat{D}A$ ..... tan chord theorem $\therefore \Delta ABC \mid\mid\mid \Delta ADB$ ..... $\angle\angle\angle$
2.2.2 $\angle BCD = 100$ ..... sum of $\angle$s in Δ $\therefore \hat{A} = 80°$ ..... $\angle$s in a cyclic quad supplementary = 180	3.1.2 $\frac{AB}{AC} = \frac{AD}{AB}$ $AB^2 = AD.AC$ ...... DA || BA alternate $\angle$'s 3.1.3 $\angle EAC = D\hat{A}E$ ..... common angle EA common side $\angle\hat{E}_1 = \hat{E}_2$ GF || EA $\therefore \Delta ACE \mid\mid\mid \Delta AED$ S$\angle\angle$
2.2.3 $\angle O_1 = 160°$ ..... angle subtended by arc or chord at the centre of circle is double the inscribed $\angle$	3.1.4 (no attempt)

Figure 4: Extracts from the mid-year Grade 12 Paper 3 examination script of L05

FINDINGS

The initial analysis of the data in the archive showed that the mean score (60%) for computational-type problems was almost double the mean score (31%) for the proof-type problems. This begins to indicate that the pupils in this pilot project have a greater aptitude for computational-type problems and is further elaborated by the 3 : 1 ratio of the pass percentage (at the 50% mark) of the computational-type to proof-type problems. It is further indicated by analysis of the computational-type answers in the archive that pupils are to some degree able to access axiomatic systems, in the way that referring to the actual objects and the related propositional statements regulates the mathematical activity. That is, pupils do draw on propositional statements to substantiate calculation arguments. Therefore, in respect of the computational-type problems, we can infer that many of pupils in the pilot study (taking into account that 70% passed at the 50% mark) are able to determine relationships between properties of geometrical figures, and to arrange arguments in an order in which each statement except the first one is the outcome of previous statements. They have achieved van Hiele's third level of geometrical reasoning, that is, informal deduction.

However, most of what is evident in the analysis of the computational-type problems appears to breakdown in the engagements with proof-type problems. The mean total score for this category of problems is 12,6 giving a mean percentage of almost 31%, and the pass percentage, at the 50% mark, is 29%. In the main, the pupils struggle to recognise and flexibly use the definitions and propositions when engaging with proof-type problems and to draw conclusions based on logic rather than intuition. This tendency was also evident in the way that the cognitive construct of proportional reasoning could not be applied in the proof-type Ratio & Proportion problems.

The analysis therefore points to the fact that, in addition to some lack of proficiency in propositional and deductive reasoning, the failure of many pupils in proof-type problems may have some other root.

CONCLUSION

The analysis revealed a greater competence in solving computational-type problems than in proof-type problems. Notwithstanding this imbalance, there was clear evidence that the majority of pupils in this cohort had achieved van Hiele's third level of geometrical reasoning, informal deduction, in the way that they could regulate the mathematical activity by referring to the actual objects and the related propositional statements. This competence is generally not reflected in the cases where proof-type problems are attempted, and is probably a consequence of pupils not similarly being able to derive appropriate sequences of statements in the absence of actual angle values and side lengths. Jones (2002: 130) ascribes this dilemma to an inability to "produce a short sequence of statements to logically justify a conclusion and to understand that deduction is the method of establishing geometric truth". Previous research has pointed to an absence of sequential development of an axiomatic system. In this regard, de Villiers (1996: 9) indicates that the van Hieles attributed failure of traditional geometry curriculum to teachers presenting the content at a level that was inaccessible to pupils. This meant that "they could not understand the teacher nor could the teacher understand why they could not understand". De Villiers also refers to Russian research concerning the 85% – 90% of Grade 5 pupils who had not attained Level 2 of van Hiele's hierarchy, attributed to insufficient geometry at primary school and exposure to Level 3 content early in Grade 6. Consequently, they developed a curriculum that ensured continuous sequencing and development of concepts. The lack of quality achievements by the pilot project pupils could in part be attributed to that fact that they were not exposed to a continuous sequencing and development of propositions deriving from an axiomatic system.

The explication of all the high school geometry content across one term of weekly teaching sessions may have been an important contributory factor in the struggle of pupils with proof-type geometrical problems. Or is it the absence of developmental sequencing of Euclidean geometry across the pupils' schooling years, or something else?

The results show that pupils were three times more successful, at the 50% mark, in computational-type problems than proof-type problems. We also observe that when they approach computational-type problems, propositions are referenced in support of computational statements, but pupils display poor competence in proof-type problems that draw on the same definitions and propositions. This disparity suggests a low level of competence on the part of pupils when explicit construction of syllogisms is required. In other words, syllogistic reasoning is condensed and masked in computational-type problems, and pupils are unable to construct valid syllogisms in proof-type problems. It appears that the propositions are used to justify and authorise computational statements, but the chain of syllogistic reasoning does not have to be mapped out. So, pupils may be doing something different and a more nuanced study is necessary to determine what exactly is going on. Is it possible that the pupils' computational work is not grounded in definitions and propositions, and that some other system may be operational?

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EFFECTIVE INSTRUCTIONAL APPROACHES USED BY TEACHERS TO TEACH GRADE 11 QUADRATIC EQUATIONS IN A CONTEXT OF SOUTH AFRICAN SCHOOLS IN LIMPOPO

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This paper discusses the effectiveness of instructional approaches to teaching used to teach Grade 11 quadratic equations. Types of errors displayed and misconceptions possessed by learners in solving quadratic equations by factoring, completing the square and using quadratic formula are identified. Four teachers and the researcher's colleague were observed teaching quadratic equations by factoring, completing the square and using quadratic formula as was the area of focus of the researcher. Questionnaires were used to interview these teachers and their five learners. The purpose of observing those lessons was to identify the types of instructions they used to teach quadratic equations by using these three methods. The intervention strategies were used by the researcher to alleviate the difficulties learners experienced in solving equations. The study is anchored within qualitative research invested in action research which has taken place in the classroom.

INTRODUCTION

The National Curriculum Statement (NCS, 2005) requires that teachers implement the curriculum effectively in the classroom. Teachers have to apply the rules and principles of NCS appropriately so that learners do not experience problems in acquiring the new learning area and the outcomes of the curriculum. However, learners are experiencing difficulties in solving mathematical problems (Center for Education and Development Enterprise, 2007). This was documented in the Trends in International Mathematics and Science Studies (TIMSS, 2003) which revealed that most learners at Grade 8 level in South Africa are not performing well in Mathematics.

Researchers (Adler, 2004; Luneta, 2008 and Setati, 2005) had documented some reasons for errors that learners display and misconceptions they possess which contributed to their poor performance. These include:

- Teachers' instructional strategies or questioning styles.
- Students' various styles of learning.
- Different levels of cognition (the concept is much higher than the cognitive level of understanding of learners), and

- Learners lack of procedural or conceptual knowledge.

RATIONALE FOR THE STUDY

The purpose of this research was to study teaching instruction in Grade 11 quadratic equations by factoring, completing the square and using quadratic formula. Errors learners displayed and misconceptions evidenced in solving quadratic equations were analysed at Mamolemane Secondary School. The problem was diagnosed through examining results in Grade 11 Mathematics, class tests, class activities and home activities in the researcher's school. In analysing the examination results, the researcher had already identified that learners had performed poorly in Mathematics for the three consecutive years of 2005, 2006 and 2007. The analysis of results also showed that learners performed poorly in quadratic equations where the researcher had expected them to score high marks. This, therefore, dropped the average percentage of Grade 11 Mathematics results to an unprecedentedly low level. The problem experienced had prompted the researcher to investigate the difficulties learners faced in solving quadratic equations along with teachers' instructional approaches used in teaching this concept.

METHODOLOGY

The study was conducted in the qualitative research paradigm as action research, which investigated problems pertinent to the researcher's work and learners' work in his school. According to Zuba-Skerrit (1996), action research is any systematic enquiry either by teachers, principals, school councillors, or any other school stake holders in teaching and learning environment that aims to gather information about ways in which their particular school operates, how they teach, and how well their learners learn.

A precondition for undertaking action research in the area of focus was for the researcher to organise and support the learners in order to produce good results. This also helped to alleviate errors and misconceptions that learners experienced in solving quadratic equations. In alleviating these errors and misconceptions, the researcher employed the intervention strategies adopted from their literature review. The intervention strategies had been used for three consecutive years, 2008, 2009 and 2010.

The researcher gave learners a pre-test to select the sample for hıs research. There were 84 Grade 11 learners who wrote the test. Learners who performed badly were the ones who were sampled to partıcıpate ın the study. The researcher dıd a self evaluatıon by teachıng learners quadratıc equatıons by usıng the stated three methods. The researcher used questıon and answer method, exposition, explanatıon, dıscussıon method, whole class and teacher led discussion. The concepts of fındıng common factors and lınear equatıons were taught before quadratıc equatıons could be introduced. The ınformed methods that were used ın 2009 and 2010 were to observe theır effectıveness ın teachıng and learning, and had allevıated the dıffıcultıes learners encountered ın solvıng quadratıc equatıons.

DATA COLLECTION

The study was anchored within a qualitative approach invested in action research, which has taken place in the classroom. Qualitative research produces descriptive data, considers people's perspectives, and occurs either in written form or verbally in its observation of the behaviour of participants (Johnson & Christensen, 2004).

Data was collected at the time of writing from four schools in the neighbourhood and the school to which the researcher was attached. Upholding the premise that data collection is the process of gathering information that helped to understand the research problem much better in order to address that particular problem, instructional approaches they used to teach quadratic equations were also considered. The researcher had visited each school three times for lesson observations. Those five teachers with their five learners per school were interviewed through questionnaires (Ary, Jacobs, & Razavieh, 2002), and learners' scripts on quadratic equations were reviewed for errors they displayed and the misconceptions they possessed. Teachers' names were used as pseudonyms, such as Teacher A, B, C, D, and E according to their respective schools, named 1, 2, 3, 4, and 5.

The purpose of lesson observations was to learn more about the instructional approaches used by those teachers in teaching quadratic equations and to compare it with the informed instructional approaches used by the researcher when conducting this research (Patton, 1980). The five teachers and their learners had participated in the research study of their own volition, and they were assured that the information they provided would be used for the purposes of the study only and under strict confidential conditions (Taylor & Bogdan, 1984).

DATA ANALYSIS

Data in this research was analysed usıng a descrıptıve narrative (Johnson and Chrıstensen, 2004). Thıs was done in order to render a descrıptıon of teachers teachıng approaches and learners work based on class tests, pre-test and post-test. The intervention strategies used by the researcher and adopted from the literature was compared and contrasted with the methods that were uninformed by the literature and used by the five teachers who participated in the study. Data analysis is the breaking down of raw data into more understandable information (McMillan & Schumacher, 1997). Analysis of the lessons was done by using Indicator Evaluation Forms for Classroom Instructions adopted from Luneta (2006). Teachers' teaching methods were referred to the question and answer method, problem solving, exposition, explanation and discussion method (group discussion, teacher-led discussion and whole class discussion). Learners' errors and misconceptions displayed in solving quadratic equations by using the three stated methods were analysed. Teachers' teaching approaches and teaching methods were analysed individually. These were as follows:

Question and answer method

This teaching method was previously not used to develop and enhance learning before the researcher adopted it from within the review of literature. Questioning is a teaching method which encourages interaction between the teacher and learners in classroom learning (Hansen, Drews, Dudgeon, Lawton & Surtees, 2006). The researcher had before the intervention strategies asked learners questions in order to get correct answers as was what other teachers had done in using this method. One had changed to choose learners randomly to identify the errors learners had in the topic concerned. The researcher wrote the equatıon on the chalkboard, $x^2 – 3x – 4 = 0$ and asked learners to gıve the type of the equation and what are the factors of the equatıon. He chose one of the learners who raısed hıs hand. This type of questioning method had disengaged other learners, especıally the slow ones, in teaching of quadratic equations and other mathematical concepts. The teachers who participated in the study had used an uninformed method of questioning learners that was focused upon obtaining correct answers, instead of developing and enhancing learning in quadratic equations (Hansen et al. 2006). This had impeded learning in quadratic equations as teachers did not identify the underlying errors and misconceptions learners possessed. Questıons lıke 'What are the factors of the equatıon?' were asked by all the teachers and answered by learners predominantly by high acheivers, who raısed theır hands. This general reticence to answer questions contributed to errors learners experienced in solving quadratic equations and hence other Mathematical concepts. The questioning technique adopted from the literature assisted the researcher in identifying and addressing the errors and misconceptions learners made in solving quadratic equations.

Telling method

The telling method involves the teacher as the main imparter of information while learners are passively listening to their teacher (Pimm & Johnston-Wilder, 2005). Previously the method was not used with care and contributed to learners errors made in solving quadratic equations and other mathematical concepts. This method was used by the researcher and the five teachers who participated in the study in teaching quadratic equations. The researcher refrained from using this method after he adopted an informed one from the literature, applying it in teaching and learning of quadratic equations and other mathematical concepts. Teacher A, B and C had used a tellıng method, but were much better as compared to D and E. Teacher D and E had done most of the talkıng and ınteracted wıth learners for a short tıme.

Pimm and Johnston-Wilder (2005) outlined two forms of the telling method as exposition and explanation. Exposition is when the teacher talks directly to learners and learners are passively listening to what their teacher is delivering to them. Explanation is when the teacher takes learners from their current knowledge position to a new one. The researcher used exposition in conjunction with explanation to reinforce understanding in the teaching and learning of quadratic equations. The researcher had previously started his lesson by saying, "Class, today I want us to treat quadratic equations by factorization and defined in the equation $ax^2 + bx + c = 0$." He then showed learners how to find factors of quadratic equations by factoring and this had contributed to errors learners made.

The observed teachers had used explanation to reinforce understanding in teaching and learning of quadratic equations except Teacher D and E. The two teachers had mostly used telling method, which showed learners how to factorise quadratic equations, how to complete the square and how to use quadratic formula, causing some learners to lose concentration during teaching and learning. This study had brought some changes, as the researcher had applied the informed approach to address the difficulties learners experienced in solving quadratic equations and other mathematical concepts. Telling method is effective when teachers use it with care in order to avoid learners losing focus during teaching and learning.

Problem solving

Problem solving is the interaction of cognitive processes and mental representation in diverse ways (Green &Gilhoolly, 2005: 357 and Halpern, 1997: 219).Luneta (2006) indicates that problem solver should understand the problem to be able to analyse it, break the problem into accessible chunks, execute the problem and reflect on the action taken. The researcher had engaged his learners in problem solving to solve quadratic equations by factoring, completing the square and using quadratic formula. Learners were given problems to solve in groups as to develop and enhance learning of quadratic equations and other mathematical problems. Problems such as $x^2 – 3x -4 = 0$ were given to learners to solve in groups and some of the learners grappled with them. Problem solving is effective when it is used in learners' group discussion as creative thinking, critical thinking and argumentation are encouraged. Teacher A, B and C had used this teaching method but were not particularly effective in solving quadratic equations by completing the square, even after the researcher had used intervention strategies. They also gave their learners problems to solve in groups except Teacher D and E. Mathematics teachers need to be encouraged to use problem solving in their teaching and learning to enhance and develop classroom learning.

Discussion method

The discussion method is the interaction either between the teacher and learners orlearners amongst themselves (Sorensen, 2003). Discussion method in teaching andlearning of quadratic equations had involved learners' group discussion, teacher-leddiscussion and whole class discussion. Previously the researcher had used groupdiscussions that were unmonitored, which resulted in the disengagement of the low achievers as high achievers were the ones who participated mostly in their groups. The researcher arranged his learners without giving them tasks to perform in their groups (Battista, 2001). This had an impact on learners' performance in quadratic equations and hence other mathematical concepts. The only way to avoid the formation of entrenched misconceptions is through discussions and interactions (Hansen etal. 2005). Teacher-led discussion was useful for learners and was considered as an important prerequisite for effective teaching (Askew, Brown, Rhodes, & William, 1997). The researcher used this type of discussion before he could use group discussion, as it gave learners directive on what had to be done in solving quadratic equations. In learners' group discussions, one had used two types of group discussions in the form of 'home' and 'family' group discussions, otherwise known as the jig-saw method (Siegler, 2003). In home group discussions learners were labelled as number 1, 2, 3, etc. and each same labelled learners form family group and solved the problem of their label so that family group 1 solved problem 1 and so on. Learners were given problems such as $x^2 + 4x = -2$ to discuss in family group and then home group. Some of the learners had regrouped the terms ınstead of addıng half the square of the coeffıcıent of *x*to solve thıs equatıon.

This type of group discussion was different from what the observed of other teachers, except for Teacher C who applied the same method of discussion. Teacher A and B had used unmonitored group discussions, which had disadvantaged other learners - especially the slow ones - as they were notably disengaged in the discussions. This was what the researcher had done before the study to teach his learners and realised that it had been ineffective after the intervention strategies adopted from the literature review. These group discussions were not used with care because learners were not given tasks to perform in their groups, such as that of scribes, reporters, leaders or captains (Battista, 2001; Sorensen, 2003). Teacher D and E had not used any group discussion whatsoever.

The learners' group discussions were imperative in leading the teacher and the learners to whole class discussion. Whole class discussion was especially appropriate towards the end of the lesson as it provided the researcher and Teacher A, B and C with a closure or conclusion of the concept (Grouws & Cebulla, 2000). In this type of discussion, teachers and the researcher had reached common agreement about how to find the factors of equations by using the first term and the last term with appropriate signs in between terms. They also concluded that in solving quadratic equations by completing the square, they added half the square of the coefficient of *x* and lastly in using quadratic formula the value of ***a***, ***b*** and ***c*** should correctly be identified with their signs per se and substituted in the formula.

Document Analysis

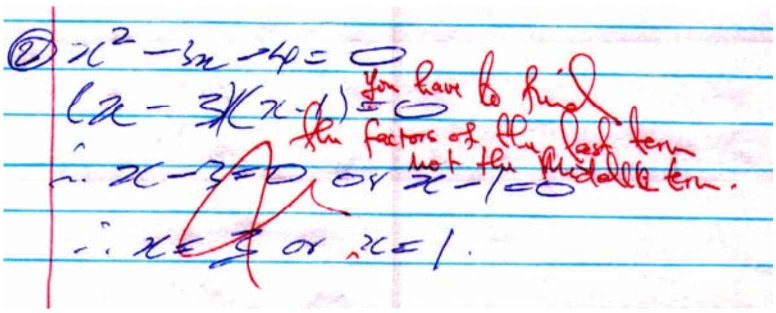
Document analysis is the process of identifying and diagnosing errors learners display in solving Mathematical problems. Errors are simple symptoms of the difficulties a learner had during learning experience (Troutsman & Alberto, 1982). Error analysis was done with the purpose of identifying the types of errors learners had displayed in solving quadratic equations by factoring, completing the square and using quadratic formula. Thisresulted in the researcher using intervention strategies and those errors and misconceptions were corrected (Mastropierie & Scruggs, 2002; Fuchs, Fuchs & Hamlett, 1994; Salvia & Hughes, 1990; 2004). The samples were taken and illustrated the types of errors learners had displayed and the misconceptions they possessed in solving quadratic equations. The researcher sampled 30 scripts for error analysis from class tests and pre-tests.

In the pre-test, most of the learners had performed poorly in solving quadratic equations by factoring, completing the square and using quadratic formula. This poor performance had been influenced by the errors and misconceptions learners possessed in solving quadratic equations by using the three stated methods. A discussion of errors and misconceptions of each method of solving quadratic equations follows.

Errors in factoring

Most of the learners experienced difficulties in solving quadratic equations by factoring. Out of 30 learners, 24 of them were unable to find factors of given quadratic equations, using inappropriate signs in between brackets and finding incorrect roots. These types of errors are caused by the inability of learners to use relevant procedures in solving quadratic equations by factoring (Kanyalioglu et al., 2003; Baykul, 1999 and Hiebert & Lefevre, 1986). Only six learners articulated the procedures of solving quadratic equations by factoring. Sorensen (2003) advises that learners had to understand concepts to be able to use the correct procedures in solving mathematical concepts. The sample below illustrates the systematic errors learners had displayed in solving quadratic equations by factoring.

Sample of quadratic equation by factoring



The researcher asked the learner why he solved the equation in that way. Below is how the learner had responded:

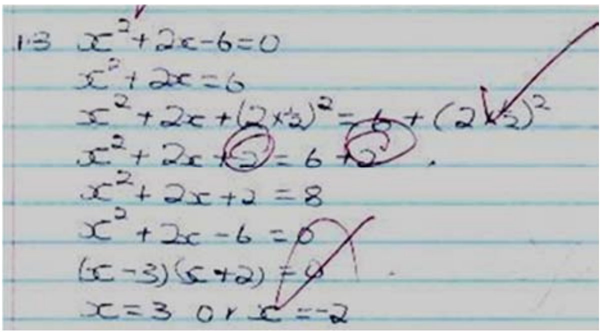
> *R: Can you explain how you solved this equation?*
>
> *L: I have found the factors of the equation by using the first term and the middle and got the factors as (x - 3) (x - 1). I added -3 and -1 to get -4 which was the last term.*

There was no correlation shown in steps used as the learner was unable to execute procedures correctly and this displayed a lack of understanding in finding factors of equations. The learner had used the wrong procedure of solving the equation by finding the first term and the middle term to factorise it. If a learner lacks conceptual knowledge, it will be difficult for him/her to articulate correct procedures to solve mathematical problems (Greeno, Riley, & Gelman, 1984; Confrey, 1990). The learner added -3 and -1 and got -4 as an incorrect answer. The learner did not understand this method of solving quadratic equations to find the solution. The equation ought to have been solved by finding the first term and the last term, that's $(x - 4)(x + 1) = 0$. Some of the errors involved the allocation of signs in the brackets, which had caused learners certain difficulty in solving quadratic equations. This indicated that learners had difficulties in solving quadratic equations by factoring, as they could have reflected back to what they had done after they got their solutions. They ought to have multiplied back to check the validity of their factors. Some of the learners were completely unable to give factors of any equations, which reflected the ineffectiveness of instruction used by the researcher before this study (Luneta, 2008). This implied that learners struggled to learn other concepts involved in quadratic equations (Battista, 2001), such as completing the square and using quadratic formula.

Errors in completing a square

Learners also experienced difficulties in solving quadratic equations by completing the square. Most learners had problems in adding half the square of the coefficient of ***x*,** dividing by the coefficient of x^2 which is not equal to 1 or less than zero and factorising the left-hand side. In adding half the square of the coefficient of ***x*,** learners had firstly squared half the coefficient of x first and multiplied it by $\frac{1}{2}$, instead of multiplying the coefficient of x by half and squaring the product. Learners failed to follow the correct procedures of solving equations by completing the square, which showed a lack of conceptual knowledge (Kanyalioglu et al., 2003; Baykul, 1999 and Hiebert & Lefevre, 1986). The following sample was taken to illustrate the types of errors and misconceptions they experienced in solving equations by using this method.

Sample of quadratic equations by completing the square

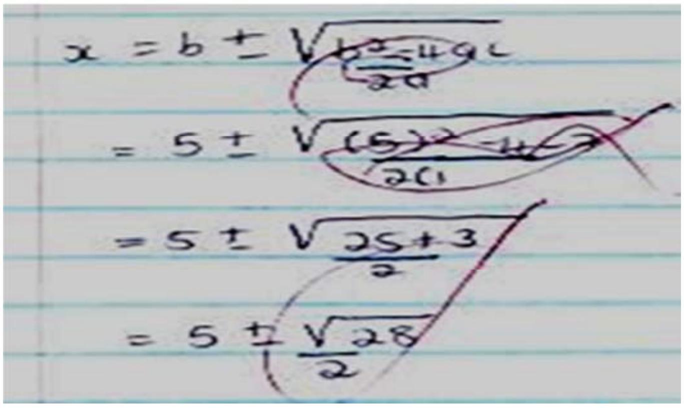


This learner displayed little knowledge of solving quadratic equations by completing the square. This had transpired when she had found the additive inverse of -6 as 6 and correctly added half the square of the coefficient of *x* to both sides. The learner had failed to simplify terms in brackets, she had squared 2 first and multiplied the product by $\frac{1}{2}$, which was incorrect as she arrived at the answer of 2. She should have multiplied 2 by $\frac{1}{2}$ and squared 1 to get the answer as 1 on both sides. The other misconception that students held was that of regrouping the constants and adding them (2 – 8 = -6). These types of errors indicated that problem-solving skills were not reinforced in the teaching and learning of quadratic equations by completing the square. The researcher used intervention strategies to try to alleviate these misconceptions, but could not seem to help, as learners continued making the same mistakes. The researcher thus investigated other strategies that might be effective to help in addressing these errors and misconceptions. Errors were also identified in solving quadratic equations by using the formula.

Errors in using quadratic formula

Quadratic formula is another method of solving quadratic equations, such as in $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Most of the learners at the researcher's school had experienced difficulties in solving quadratic equations by using quadratic formula before the study. Some of them were unable to write the equation in standard form, $ax^2 + bx + c = 0$. 25 learners out of 30 who were sampled were unable to write the equations in standard form, such as $2x^2 + 8x = -3$, where they incorrectly identified the value of *a* as 2, *b* as 8 and c as -3. The wrong formula was used and incorrect substitution was also used. The other problem was in simplifying terms in the square root, which resulted in learners getting incorrect answers in solving quadratic equations by using the formula. The sample below illustrates some of the errors learners displayed in solving equations by using quadratic formula, the equation was $x^2 - 5x - 7 = 0$.

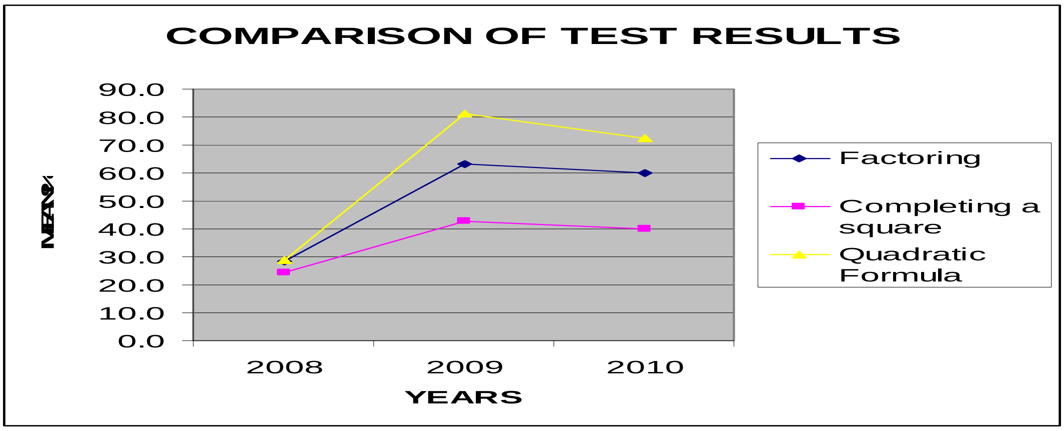
Sample of quadratic equation by using quadratic formula



The sample was taken from one of the learners in Grade 11 class. This learner had used the wrong formula. The formula should have included all the numerators divided by 2*a*, not terms in the square root divided by 2*a* and *b* should be negative. This displayed a conceptual error, which resulted in following the incorrect procedure of solving quadratic equations by using quadratic formula. The learner had also used incorrect substitution of the value of ***a, b*** and ***c*** in quadratic formula, because he did not consider the negative signs of the terms like -5 and -7. When multiplying terms - especially in conjunction with more than one sign, brackets have to be introduced in the multiplication of terms to give the expression meaning like the ones in the square root. The substitution should have been $x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-7)}}{2(1)}$. In the sample he added -4 to -7 and arrived at 3, which was incorrect application of the two signs. The correct answer is -11. The value of *a*was not substituted in the formula which was important to substitute it in the formula as marks are allocated for correct substitution in tests and examinations. The equation should have been solved by having *x* as, $x = \frac{5 \pm \sqrt{25+28}}{2}$, which ought to have given him $x = \frac{5 \pm \sqrt{53}}{2}$. He could have left his answer at this point, which is in surd form or could have used the calculator to correct it to one or two decimal places, such that *x* = 6,3 or -1,3. This in turn reflected the ineffective approaches used to teach this concept (Luneta, 2008).

These types of errors displayed and misconceptions possessed by learners in solving quadratic equations by using the formula were addressed by the researcher, who used intervention strategies adopted from a body of relevant literature. The researcher emphasised that learners understand how to write the equation in standard form $2x^2 + 8x = -3$, to be $2x^2 + 8x + 3 = 0$. Learners were given more equations that were not written in standard form for them to solve, so that they could be exposed to them. Learners had turned to be able to use the correct formula and had written the equations in standard form ($ax^2 + bx + c = 0$). The value of ***a***, ***b*** and ***c*** were correctly identified and substituted in quadratic formula. These interventions had played a major role in alleviating the difficulties learners had had in solving quadratic equations by using the formula. The mean percentage of solving quadratic equations by using quadratic formula after intervention strategies was much higher when compared to factoring and completing the square.

The researcher had used the graph below to compare the results of pre-test with the short tests given to learners according to the stated sub-topics of quadratic equations. The pre-test results were also analysed separately, according to the sub-topics reflected in so-called short tests. The graph that follows illustrates the comparison of tests results written in 2008, 2009 and 2010, created to check the validity of the intervention strategies the researcher had used in teaching quadratic equations. The graph compares the test results for the year 2008, 2009 and 2010 based on factoring, completing the square and using quadratic formula.



The graph above had shown that the intervention strategies used by the researcher had played a role in teaching quadratic equations, except in completing the square which had not made much difference in 2009 and 2010. There was a satisfying shift in learners' percentages in solving quadratic equations by factoring and using quadratic formula. In factoring, most learners were able to find factors of quadratic equations, allocated appropriate signs in between terms in brackets and had given the correct roots or solutions. Learners had also understood how to write the equation in standard form ($ax^2 + bx + c = 0$), where previously others did not write them in standard form especially when they applied quadratic formula. Learners were given the problem $x^2 = 12 – 9x$ and most of them identified the value of ***a***, ***b*** and ***c*** without writing it in standard form, $x^2 + 9x – 12 = 0$, before they could solve it. Since the intervention strategies were applied by the researcher, they ensured that they have written the equations in standard form before they could identify the value of ***a***, ***b*** and ***c***. They had correctly used the formula, substituted the value of ***a***, ***b*** and ***c*** in the formula and had successfully solved the equation to get the solution. Most of the learners still experienced difficulties in solving quadratic equations by completing the square and this, therefore, showed the researcher that the intervention strategies were ineffective in teaching this concept. Learners were still unable to add half the square of the coefficient of *x* both sides, on the left hand side and right hand side of the equation.

CONCLUSIVE STATEMENT

Teachers that were not informed by the literature used to teach quadratic equations by factoring, completing the square and using quadratic formula had contributed to learners' errors and misconceptions. These types of instructions were deemed to be ineffective in teaching quadratic equations by using the three methods and other mathematical concepts. Intervention strategies such as questioning, problem solving, exposition and explanation, as well as discussion methods that had been employed by the researcher, had played a major role in alleviating the difficulties learners experienced in solving quadratic equations. They had previously not used them with care, which contributed to learners' errors and misconceptions in solving quadratic equations and other mathematical concepts. These informed approaches had been used in 2008 during remediation and were effective in 2009 and 2010 in teaching quadratic equations and other mathematical concepts. However, these methods were ineffective in solving quadratic equations by completing the square, as the learners still experienced problems in solving these types of equations. The intervention strategies for teaching quadratic equations by completing the square still needed to be investigated to alleviate learners' difficulties in this concept. It will be imperative for teachers to use these intervention strategies to teach mathematical concepts in their schools to enhance and develop teaching and learning. Research still has to be conducted in other methods of solving quadratic equations such as solving quadratic equations by squaring both sides, solving equations with fractions, solving equations by substitutions and solving simultaneous equations.

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EXAMINING THE USE OF BERNSTEIN'S NOTION OF *CLASSIFICATION* IN MATHEMATICS EDUCATION RESEARCH

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This paper interrogates how Bernstein's concept of classification, the strength of the boundary between categories, has been operationalized in empirical research studies in mathematics education. The paper demonstrates that the notion of classification with respect to mathematics and the everyday, as employed within Bernsteinian research, focuses on reference to everyday objects rather than the objects which emerge through evaluation and provides limited insight into the nature of school mathematics realised in pedagogic contexts. Building on work done within the problematic of the constitution of mathematics, we develop a more mathematically-attuned notion of classification as a methodological resource for analysis of mathematics constituted in pedagogic contexts.

INTRODUCTION

With the outcome of each national test or examination we are reminded of the crisis in South African education. The crisis, which manifests itself in numerous ways, is most evident in the disparity in performance between schools populated by middle-class students and those by working-class students. Student performance in national and international assessments are still largely differentiated along social class lines. However, social class remains intertwined with race in South Africa where the working-class is largely 'African' and 'coloured' and the middle-class still largely white. In the Western Cape, only 9,1% of candidates in ex-DET schools compared to 81,1% of candidates in ex-CED schools scored 50% or more in the 2008 National Senior Certificate Mathematics examination, the first examination of the FET National Curriculum Statement (Personal communication, Jon Clark, April 2011). Social class, and so too race, therefore continues to be strong predictors of success in school mathematics.

Differences in performance along social class lines is not a new issue, and has received considerable attention in the educational sociology literature (Cooper & Dunne, 1998; Cooper, 2000; Dowling, 1998; Taylor, 1999; Muller & Taylor, 2000; Lubienski, 2004; Cooper & Harries, 2005; Hoadley, 2007). In particular, research studies using Bernstein's sociological theories of education, have based their arguments on either one or both of the following propositions: the first, drawn from Bernstein's code theory, posits that working-class children are predisposed towards context-dependent orientations to meaning or making sense of the world which differs from the context-independent orientation to meaning required of formal schooling in general and mathematics in particular; and the second proposition, based on Bernstein's theory of pedagogic discourse, asserts that the form of knowledge transmitted and acquired in working-class schools is incongruent with the form of knowledge required for success in schooling. Some studies (see Cooper, 2000; Cooper & Dunne, 1998; Lubienski, 2004;) focus solely on the first proposition whereas other studies (see Hoadley, 2007; Taylor, 1999) consider both propositions as resources to explain the achievement gap between middle-class and working-class students in school mathematics.

The central idea linking both propositions is Bernstein's concept of *classification* which refers to

relations between categories, these relations being given by their degree of insulation from each other. Thus strong insulation created categories, clearly bounded, with a space for the development of a specialised identity, whereas the weaker the insulation, the less specialised the category. (Bernstein, 2000: 99)

Classification in Bernsteinian research, with respect to learners' orientation to meaning, refers to the predisposition of working-class students to misrecognise the classificatory principle of the schooling context. In other words, they claim that such students fail to recognise the specificity of school mathematics and tend to use knowledge of everyday contexts in school mathematics contexts. In relation to knowledge distributed in pedagogic contexts, much Bernsteinian research uses what they refer to as classification with respect to (w.r.t) the relationship between school mathematics and everyday[20]. The research of Cooper and colleagues on learners' responses to mathematics test items and Hoadley's research on the comparison of the pedagogy in middle-class and working-class schools are good examples of Bernsteinian research referred to above.

Below the research of Cooper and colleagues and Hoadley's research is examined in terms of how the concept of classification is used.

COOPER & DUNNE: 'REALISTIC' TEST ITEMS

The work of Barry Cooper and his colleagues focuses on examining the relationship between students' performances on mathematics test items and their social class origin and gender. One of the conclusions drawn by Cooper and his colleagues is that working-class students performed less well than middle-class students on 'realistic' test items. 'Realistic' items, defined by Cooper as items which "contains either persons or non-mathematical objects from 'everyday' settings" (Cooper & Dunne, 2000: 84) are contrasted with esoteric mathematics items[21]. A test item that involves finding the price of a box of popcorn given two bits of information: (1) a coke and a box of popcorn cost 90p and (2) two cokes and a box of popcorn costs £1.45 is

[20] Other uses of classification is not dealt with in this paper.

[21] Here Cooper & Dunne appear to draw on Dowling's (1998) notions of esoteric and public domains. Given the limited space, a discussion of Dowling's domains practice is omitted from this paper.

classified as a 'realistic' item where according to Cooper & Dunne the boundary between mathematical knowledge and everyday knowledge is weak i.e. the item has a weak classification w.r.t mathematics and the everyday. The test item, "n stands for a number. $n + 7 = 13$. Find the value of $n + 10$" is an example of an esotoric mathematics item where the classification between mathematics and the everyday is strong (Cooper & Dunne, 2000: 84-85).

Since evaluation reveals criteria for the recognition and realisation[22] of mathematical objects or procedures in pedagogic contexts (Bernstein, 2000; Davis & Johnson, 2007), an examination of the evaluative criteria inherent in the coke and popcorn item reveals the objects required for solving the test item. The problem is to determine the price of popcorn (and so, too, the price of coke). The solution to this problem can be solved through (1) setting up and solving two equations simultaneously or (2) syllogistic reasoning based on the cost of one coke and one box of popcorn and two cokes and one box of popcorn.

Solution 1: If the cost of coke is C and the cost of popcorn is P, then we have two equations $C + P = 90$ and $2C + P = 145$. Solving the two equations simultaneously, produces the solution $C= 55$ and $P = 35$. So a coke costs 55p and popcorn 35p.

Solution 2: If a coke and box of popcorn cost 90p,

and a coke and a coke and a box of popcorn cost £1.45 (145p),

then a coke must cost 55p.

So a box of popcorn must cost 35p.

The solutions show that the item requires, at times, the recognition of the objects *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn* as single objects rather than as two and three distinct objects, respectively. Values are assigned to *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn*, and it is the value of the object *popcorn* that has to be calculated.

The objects which emerge here are related through particular mathematical operations and therefore constitute particular mathematical relationships. The classificatory principle intrinsic to the problem is therefore located solely in mathematics and not the everyday. The coke and popcorn item is categorised as weakly classified w.r.t mathematics and the everyday by Cooper & Dunne because it contains references to everyday objects. Similarly, the second item is referred to as an esoteric mathematics item because it does not contain any references to the everyday. Cooper & Dunne's notion of classification w.r.t to mathematics and the everyday is based on *reference* to everyday objects rather than *evaluation* which reveals the objects circulating in the pedagogic context.

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[22] Although Bernstein links the recognition rule to classification and the realization rule to framing using 'what' and' how' as overarching themes in his work, here 'recognition' and 'realisation' are used very loosely to refer to what knowledge is acquired and how this knowledge is acquired.

The response of a working-class student to the coke and popcorn item is shown below.

I said to myself in a sweetshop a can of coke is normally 40p so I thought of a number and the number was 50p so I add 40p and 50p and it equalled 90p. (Cooper & Dunne, 2000: 41)

The student's response to the 'realistic' test item is typical of working-class students according to Cooper & Dunne. Their explanation for the student's response proceeds as follows: the student recruits her everyday knowledge of shopping as a resource to solve the problem and does not recognize the system of simultaneous equations implied by the problem. The student fails to recognise the specificity of the context, blurs the boundary between school mathematics and the everyday context of shopping, thereby weakening the classification w.r.t mathematics. They claim that the student responds in a 'realistic' rather than 'esoteric' manner to the item (Cooper & Dunne, 2000: 199).

However, in subsequent research, Cooper & Harries (2005) found that it was working-class students who had greater difficulty than middle-class students in solving simple 'realistic' word problems, particularly the calculational type problems such as the 'lift item'. The particular item referred to a sign, situated in an office block lift, which stated that: "The lift can carry up to 8 people". The task was to calculate how many times the lift "must go up" to transport 76 people during morning rush hour. The problem assumes that the lift is always full and that everyone uses the lift.

Cooper & Harries (2005) found that many working-class students students correctly calculated 76 divided by 8 to produce the answer 9,5 but they ignored the fact that the problem referred to *lift trips* and so failed to see that the answer should be 10 trips. Here, according to Cooper & Harries, there is an expectation that the students would weaken the classification w.r.t mathematics and the everyday, but it is the working-class students who create strong boundaries between mathematics and the everyday. On the one hand, according to Cooper and colleagues, the failure of working-class students is located in their recruitment of the everyday (weakening classification w.r.t. mathematics) and, on the other hand, their failure is located in their suspension of everyday considerations (strengthening classification w.r.t. mathematics). So, what is it to be?

Cooper & Harries (2005) recognise the anomaly in their findings. They argue that for the short calculational 'realistic' word problems (like the lift item), in contrast to more extended context problems (like the coke and popcorn item), the presence of numbers prompts learners to perform calculations and to ignore the everyday. It is curious that the classificatory boundary is able to shift depending on the type of problem presented to students. This is even more curious given that Cooper and colleagues base their arguments on the predisposition of working-class children to use everyday knowledge inappropriately when solving mathematics test items (Cooper & Dunne, 1998: 125).

This relative failure (and it is relative, not absolute) to recognise the strongly classified nature of school mathematics in the face of surface appearances which suggest everyday knowledge may be an aspect of sociocultural predispositions discussed by Bourdieu and Bernstein (Cooper & Dunne, 1998:140)

This paradox in Cooper and colleagues' findings renders their deployment of the concept classification w.r.t mathematics and everyday knowledge inconsistent because if they base their arguments on the predisposition of working-class students, then this predisposition shoud be consistent and should explain their data.

It seems that the working-class students in Cooper & Dunne's study recognise *coke*, *popcorn* and *money* as objects which are immediately recognisable in the coke and popcorn item but fail to recognise the more complex objects *coke*$\wedge$*popcorn* and *coke*$\wedge$*coke*$\wedge$*popcorn*, which are central to solving the problem. The syllogistic reasoning of the working-class student described above is as follows: if I have the value associated with *coke* then I can find the value associated with *popcorn*. Since the value associated with *coke* is not provided in the test item, the student derives the value associated with *coke* from experience and is then able to find the value associated with *popcorn* by calculation.

In the lift trip item, the *total number of people, maximum number of people in the lift at one time* and a *lift trip* are the objects required to be recognised. The *total number of people* and the *maximum lift capacity* are easily recognisable as objects from the question but the *lift trip* is a less obvious object. It seems that the working class students recognise the quantities provided and recognise that these have to be operated with. Some of these students choose the correct operation, division, but neglect to convert the fractional answer to a whole number because they fail to recognise that they have to calculate the number of *lift trips*.

In both test items, working-class students recognise that they must find suitable objects with which to calculate and to relate these objects appropriately. However, it seems that the students' failure to produce correct solutions to the problems can be traced to their inability to select the appropriate objects from the range of objects available in the test items rather than to decisions about whether or not to recruit everyday knowledge. In other words, the students fail to grasp the classificatory principle encoded in the test items. Again Cooper and collegues' analysis of students' responses to 'realistic' mathematics items is based on reference or lack of reference to the 'everyday' rather than the objects constituted by the students' criteria for solving the problem.

HOADLEY: TEACHING SCHOOL MATHEMATICS

Hoadley (2007) uses Bernstein's categories of *classification* and *framing* together with Dowling's concepts of *localising* and *specialising strategies* to describe variations in pedagogic modalities in different social class contexts in schools. Hoadley (2007) considers classification in terms of three dimensions: 1) relations between discourses (relations between the subject area and other subject areas; and relations between the subject area and everyday knowledge); 2) relations between spaces; and 3) relations between agents. Only classification w.r.t relations between mathematics and the everyday is considered here. Hoadley (2007) concludes that the relationship between everyday knowledge and school knowledge is dealt with differently in different social class contexts. She describes mathematics of the working-class schools in her study as weakly classified because of the incorporation of everyday knowledge, and mathematics lessons in the middle-class schools as strongly classified w.r.t mathematics (Hoadley, 2007: 704).

Below we examine Hoadley's operationalization of classification w.r.t school knowledge and everyday knowledge in relation to a lesson conducted by a Grade 3 teacher in her study (Hoadley, 2007: 688). The teacher starts the numeracy lesson by reading a word problem from a textbook. A transcript of an extract from the lesson illustrates how the teacher engages the learners in repeating a sentence she reads from the textbook in a chorus-like fashion:

Teacher: Listen, on page 63, how a tree lives and grows. It says that …what does it say people. How a tree lives and grows. What does it say?

Learners: How a tree lives and grows.

Teacher: What does it say?

Learners: How a tree lives and grows.

Teacher: What does it say?

Learners: How a tree lives and grows.

The chorusing is followed by the teacher translating the sentence into isiXhosa and the learners repeating the sentence in isiXhosa followed by chanting the sentence in English again. The teacher and the learners deal with the rest of the word problem—"Pulani has about 289 trees on her farm. Write the number of trees to the nearest hundred"—in the same way. After reading the word problem, the teacher draws two trees on the board and talks about trees being shaped differently. The teacher writes the number '79' on the board for learners to write to the nearest hundred.

When the learners are unable to solve the problem, the teacher states a rule which the learners chorus:

Teacher: I say to you if the number is over 50 then it's a 100, if it's over 100 then it's 200, if it's over 300 …

Learners: [chant] Then it's 400, if it's over 400 then it's 500.

The teacher records six numbers on the board that the learners read from the textbook. She asks them to write the numbers to the nearest hundred. When a learner rounds–up 114 to 200, the teacher asks the researcher whether the learner is correct.

The researcher responds that it is incorrect. The teacher then provides a revised rule for rounding to the nearest hundred.

Teacher: You haven't started writing. I want the nearest hundred. Write. I said to you if it is over 50 the nearest hundred, it goes to hundred. If it is below 50 it doesn't go to a 100. If it is above 150 something it goes to 200. If it is below 150 something then it doesn't go. Do you understand? Same as if it's above 200 and something. If it is 250 something and above it goes to 300, if it doesn't go above 250 something, then it doesn't go to 300. It remains 200, ne? Do you understand? We are going to explain it again tomorrow.

By the end of the lesson the learners copy the word-problem and the numbers from the board but do not successfully complete the problems. The teacher does not return to this exercise later, or on the following day.

According to Hoadley (2007: 688) "the semantic resources for the lesson lay in everyday knowledge" and not mathematical knowledge. She coded the lesson as very weakly classified because she saw the boundary between school knowledge and everyday knowledge as being very weak.

If we examine the teacher's lesson more closely, we notice that the teacher focuses on three different aspects in the lesson: 1) reading English and translating from English to isiXhosa; 2) shapes of trees; and 3) 'rounding' numbers. The lesson can therefore be partitioned into three segments, each with its own focus. In the first segment, the teacher reads what appears to be the title of a section "How a tree lives and grows" from the textbook, asking the learners to read the sentence repeatedly. It is not clear from Hoadley's description whether the learners read the sentences or simply repeat what the teacher says. The teacher then translates the sentence into isiXhosa and asks learners to repeat what she says. In this segment the teacher appears to be doing 'literacy' before doing mathematics. In the second segment the teacher discusses shapes of trees with learners. This segment is very short with the teacher stating that trees have the shape of umbrellas or circles. When learners are asked to provide other shapes, they merely repeat what the teacher had said. The first and second segments of the lessons take about 24 minutes of the 35 minute lesson. In the third segment, the teacher deals with rounding numbers to the nearest 100.

The lesson as a whole is fragmented with the link between the segments being the word problem involving trees and less than a third of the time is spent on rounding numbers to the nearest hundred. That trees seem to be the organising principle of the lesson is true, but that 'trees' serve as the 'semantic resources' for the lesson is questionable. What does it mean to use the everyday knowledge of trees in a school mathematics lesson? Everyday knowledge of trees, for example that a tree has leaves or provides shade, is not used in the lesson. When the teacher eventually deals with rounding numbers, she is no longer referring to trees.

In fact the teacher changes the problem, strengthening the classification between mathematics and the everyday in Hoadley's formulation of classification. The original question refers to the object *a collection of trees*, with which is associated an approximate value, 289. So the idea, in “everyday” terms, seems to be: if Pulani has about 289 trees then we can say that she has about 300 trees. Here, however, the bald 79 of the teacher no longer refers to an amount of everyday objects. By focusing on the computational requirement of the problem the teacher attempts to fashion a procedure for rounding and she ejects the “context”.

Let’s examine the teacher’s first rule for rounding numbers to the nearest hundred: “I say to you if the number is over 50 then it’s a 100, if it’s over 100 then it’s 200, if it’s over 300 …”. Her rule (where n is the number to be rounded to the nearest 100) can be written as follows: If $n > 50$, then $n = 100$

If $n > 100$, then $n = 200$

If $n > 200$, then $n = 300$

If $n > 300$, then $n = 400$; and so on ….

We observe that the teacher has an explicit procedure for rounding-up numbers to the nearest 100 and it is a procedure which the learners grasp inductively. The teacher however reconfigures the procedure after a learner rounds-up the number 114 to 200, a correct response according to the teacher’s original procedure.

I said to you if it is over 50 the nearest hundred, it goes to hundred. If it is below 50 it doesn’t go to a 100. If it is above 150 something it goes to 200. If it is below 150 something then it doesn’t go. Do you understand? Same as if it’s above 200 and something. If it is 250 something and above it goes to 300, if it doesn’t go above 250 something, then it doesn’t go to 300. It remains 200, ne?

The teacher’s revised procedure can be written as:

If $n > 50$, then $n = 100$

If n > 150, then n = 200

If $n > 250$, then $n = 300$

Here the teacher introduces a procedure for rounding-up and it is only when she says “if it doesn’t go above 250 something, then it doesn’t go to 300. It remains 200, ne?” that you realise that she also has a procedure for rounding-down numbers to the nearest 100. It is highly unlikely that the students pick up on this criterion. There is no doubt that Hoadley is correct in her overall conclusion that the teacher creates confusion about rounding numbers to the nearest 100, that the topic is not satisfactorily dealt with in the lesson and that about two thirds of the lesson is spent on non-mathematical activity of reading and translating a couple of sentences from a textbook. But contrary to Hoadley’s analysis, this section of the lesson focuses entirely on mathematics with no reference to everyday objects or contexts. As such, the boundary between mathematics and the everyday, in Hoadley’s terms, seems to be strongly classified.

Does it mean that when the teacher is reading the word problem that contains the word ‘trees’ that she is focusing on ‘trees’ as everyday objects or when she is describing the shapes of trees that the topic is 'trees'? So what is meant by 'everyday knowledge' and how is the use of 'everyday knowledge' recognised empirically? Hoadley (2007) uses Bernstein's distinction between educational knowledge and everyday knowledge where educational knowledge is defined by Bernstein as "uncommonsense knowledge […] knowledge freed from the particular, the local" (Bernstein 1971: 215). It appears that for, Hoadley, it is ***references*** to extra-mathematical objects rather than ***knowledge*** of the objects, in this case trees, that serve as indices of context-dependent meanings and so 'everyday knowledge'. 'Everyday knowledge' appears to be a catch-all term for extra-mathematical referents or even extra-topic referents.

Unlike the research conducted by Cooper and colleagues, Hoadley focuses on evaluation in pedagogy. She measures the criteria generated through evaluation in terms of clarity and explicitness rather than on the objects which emerge in the pedagogic context. The evaluative criteria pertaining to the lesson extract involving rounding numbers to the nearest hundred is coded by Hoadley as unclear and implicit. However, the teacher's first rule for rounding-up numbers to the nearest 100 is clear and explicit; it is a rule which the learners grasp. When the teacher restates the rule in a different way, the rule is explicit although it is mathematically imprecise since the teacher says "if it is above 150 something" probably meaning if the number is greater than 150.

How are we to make sense of Hoadley's analysis of the Grade 3 lesson? Her analysis only makes sense if we read 'explicit evaluative criteria' as criteria that resonate with the mathematics encyclopaedia, in other words, mathematically correct criteria and not the evaluative criteria actually present in the pedagogic context. Hoadley's deployment of Bernstein's concept of classification with ***reference*** to everyday objects as the index of classification and the focus on the clarity and explicitness of the evaluative criteria is of little value in revealing the content of the classificatory principle and addressing what is constituted as mathematics in pedagogic contexts. Hoadley and Cooper both seem to be using a notion of classification that is much closer to Dowling's (1998) use of classification where the focus is on reference to everyday or esoteric mathematics objects.[23]

REVISITING CLASSIFICATION

Bernstein (2000) provides us with a robust theory of pedagogy. His concept of classification remains useful but has not been utilised in ways that are beneficial for analysing the contents realised in pedagogic contexts. The concept of classification

[23] Dowling is used by Hoadley (2007), Sethole (2007) and others to overcome what they perceive as the limitations of Bernstein's classification concept. Dowling they claim provides a means for analysing the contents of classification.

marks out categories that are sets or collections of objects. Within set theory, two different conceptions of collections or sets exist, an intensional definition of a set and an extensional one (Maddy, 1988). An intensional set defines a rule of membership. So members of a set belong to a set based on a membership rule.[24] In contemporary set theory, the preferred notion of a set is an extensional one because:

it is simpler, clearer, and more convenient, because it is unique (as opposed to the many different ways intensional collections could be individuated), and because it can simulate intensional notions when the need arises … (Maddy, 1988: 484)

The axiomatic basis of the extensional notion of sets is based on the Zermelo-Fraenkel axioms, particularly the axiom related to extensionality. In the formal language of the Zermelo-Fraenkel axioms, the axiom of extensionality can be stated as follows:

$$\forall x \forall y \forall z[(z \in x \Leftrightarrow z \in y) \Rightarrow x = y)]$$

In other words, if two sets x and y have the same elements then they are identical. The axiom of extensionality is "not just a logical necessary property of equality but a non-trivial statement of belonging" (Halmos, 1974: 2).

Bernstein's notion of classification, particularly his definition of classification in terms of a boundary which distinguishes one set or collection from another can be considered extensionally, even though he may not have considered the notion of classification in terms of extensionality. In other words, the extension of the category, that is the members of the category reveals the boundary and so the classificatory principle. Bernstein's definition of classification does not specify what the nature of the objects within the boundary should be since he is agnostic about the contents within the boundary. Bernstein's agnosticism regarding the nature of the objects aligns his definition with mathematics which similarly ignores the nature of objects. In mathematics, a set can consist of any objects Whether these objects are dogs, numbers or even sets is, in a general sense, immaterial. This leads us to consider developing a mathematically-attuned notion of classification.

CONCLUDING REMARKS: MATHEMATIZING *CLASSIFICATION*

Pedagogy is understood as fundamentally evaluative (Bernstein, 2000). Evaluation distinguishes legitimate from non-legitimate knowledge statements for learners and reveals criteria for the recognition and realisation of mathematical objects or procedures in pedagogic contexts. Therefore, the regulation of *what* comes to be

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[24] For example, members of the ANC political party is an intensional set because a member of the ANC is determined on the basis of a membership card, the membership rule. An extensional set, on the other hand, is defined by the members of a set. For example, the set of cards dealt randomly to a player by a card dealer in a game is an extensional set. The set consists of the cards received from a dealer where the set is not determined by a membership rule.

constituted as mathematics and *how* mathematics is constituted in a pedagogic context is rendered visible through the criteria that circulate in that context (Davis & Johnson, 2007).

The question of the constitution of mathematics in pedagogic contexts implies an extensional notion of sets for the observer. The classification principle operative in a pedagogic context is determined by the nature of the objects which emerge through the evaluative criteria circulating in a particular pedagogic context. Therefore what is constituted as mathematics and how it comes to be constituted reveals the classification principle functioning in a particular pedagogic context.

Davis' (2010) analysis of the teaching of addition of integers in a Grade 10 class provides a useful example for illustrating how we might ascertain the classificatory principle operative in this pedagogic context. His analysis proceeds as follows: the problem, calculate -7 + 2, at first glance implies that the objects concerned involve addition over the integers. In other words, the problem appears to involve the magma $(\mathbb{Z}, +)$. The teacher's procedure for solving the problem is provided in the extract below.

Teacher:	So if the signs are the same .. what do you do? .. You take the common sign .. and then .. you add. … If the signs are not the same .. what do you do? You subtract.
Learners:	[Chorus.] Subtract.
Teacher:	But first you take the sign of the what? .. The sign .. of the bigger number. You look at the bigger number between the two .. and then you take the sign .. of the bigger number.
Learners:	[Chorus.] Yes.
Teacher:	This should always be the case. (School P6, Grade 10, February 2009.)

Davis' (2010) analysis of the operational activity of the teacher reveals that the teacher's procedure involves a shift from the object $(\mathbb{Z}, +)$ to $(\mathbb{N}, -)$, that is to subtraction over whole numbers, and the use of a series of auxiliary operations and operation-like manipulations such as detaching signs from numbers (sundering) and joining signs to numbers (concatenation). His analysis illustrates that if you adopt an intensional stance, then the classificatory principle appears to be knowledge of the operatory properties of $(\mathbb{Z}, +)$. However, an extensional approach to determining the objects populating the pedagogic space reveals quite different objects from that encyclopaedically indexed by the signifiers of the problem.

The operational activity of the teacher provides insight into the existential specificity of the objects which emerge in the pedagogic context and the logic of how these objects relate to each other. However, over and above the analysis of operational activity of teachers and learners, Davis (2011) alerts us to considering what regulates the existential nature of the primary objects of focus in pedagogic contexts. His notion of *ground* provides insight into the existential decisions made by teachers and learners.

The categories [of ground] were intended to draw out and name the over-determining effects on mathematical activity of the fixing of the primary objects of attention through existential decisions about what they are *qua* mathematical stuff (Badiou, 2006), and as carried by pedagogic evaluation (Bernstein, 1996; Davis, 2005). (Davis, 2011)

In the integer example above, it is the procedure for performing arithmetic operations on integers that regulates the operational activity of the teacher rather than the operatory properties of the magma $(\mathbb{Z}, +)$. The form of ground operative in this pedagogic context is therefore *algorithmic* and not *propositional* (see Davis (2011) for an in depth discussion of forms of ground).

The above example and others discussed in Davis (2010; 2011) and Jaffer (2010; 2011) enable us to redefine *classification* as the properties of magma and magma-like objects that emerge from an analysis of the operational activities of teachers and learners in pedagogic contexts and an analysis of the form of ground which regulates the operational activity of teachers and learners. *Classification* refers to the domain of objects operated over and the operatory logic at play in the pedagogic context. The principle of classification is ascertained through an analysis of evaluation which reveals the form of knowledge constituted and distributed in pedagogic contexts.

The notion of classification developed here has the potential for providing an analytic resource enabling a more nuanced descriptions of mathematics pedagogy in different social class contexts that go beyond the focus on the academic/everyday distinction and social relations in pedagogic contexts. Application of these concepts are required in middle-class schooling contexts to assess the potential of these analytic resources for analysing the disparity in learner performance in different social class contexts. The latter remains a project to be pursued.

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AN APPROACH INFORMED BY SOCIO-CULTURAL THEORY TO LEARNING OF DERIVATIVES IN A UNIVERSITY OF TECHNOLOGY

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This paper uses an approach, which is informed by socio-cultural theory and focuses on the zone of proximal development to explore students' understanding of the rules of differentiation. The study on which this article is based investigated specific ways of social interaction that can advance students' understanding from an elementary level to a higher level in differentiation. The researcher used a qualitative case study approach and collected data from students' written work, observations, audio-visual recordings, and in-depth interviews. The students displayed problems in manipulation of derivative of trigonometric functions. The use of ZPD as a framework to develop a teaching approach towards the improvement of students' understanding showed that the students developed positive self-esteem.

INTRODUCTION AND BACKGROUND TO THE STUDY

Several researchers argue that in spite of many attempts and a variety of approaches that have been adopted to improve students' understanding of derivatives, the problem of poor performance persists with first-year university students (Barnes, 1995; Naidoo & Naidoo, 2007; Tall, 1985, 1992). Lerman (1996) therefore, proposes implementation of an approach which is informed by socio-cultural theory of learning in the teaching and learning of mathematics, but there are few studies that focus on exploring the use of this kind of approach to improve students' understanding of differential calculus.

The researcher worked with students who were registered for mathematics in the extended curriculum programme, which entails a curriculum that is designed for students who are borderline cases who nearly meet the minimum academic requirements for admission to the main engineering stream, but show potential based on psychometric testing to succeed in their studies. The minimum requirements are that the students should have obtained at least 50 per cent in Mathematics, Physical Science and English in the matriculation examination, as well as entrance to a university of technology.

Students registered in extended curriculum programme study the same content of Mathematics as their counterparts in the main engineering stream, but instead of learning it within a semester, they learn it over a period of one year.

The aim of this article is to discuss the use of a teaching approach which is informed by socio-cultural learning theory focusing on the Zone of Proximal Development

(ZPD) and the More Knowledgeable Other (MKO). A further aim is to indicate how social interaction has increased students' insight into rules of differentiation and helped to advance them from an elementary to a higher level of understanding.

RESEARCH QUESTION

To what extent can a socio-cultural learning approach contribute to promoting students' understanding of differential calculus?

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

In South Africa, first-year university Mathematics students show poor understanding in their learning of Mathematics in spite of many attempts to improve their understanding of the subject (Wolmarans, Smit, Collier-Reed & Leather, 2010). Some difficulties associated with students' poor performance in calculus concern their memorisation of rules and procedures without making sense of the basic concepts as a prerequisite to master calculus (Mofolo-Mbokane, 2010). Wolmarans ***et al.*** (2010) argue that "the weaknesses in the school Mathematics examinations are directly responsible for the poor performance in Mathematics 1 at university level" (p.275).

In order to address the large number of underperforming students in first-year university Mathematics, the researcher thought of a socio-cultural approach. Socio-cultural theories provide a direct challenge to a constructivist approach as they argue that adults and lecturers in particular, should be involved in developing students' understanding and in so doing, should leave their mark on what students learn (Brodie, 2010). This research employed a socio-cultural theoretical framework to encourage a student-centred and activity-based approach to develop understanding in the learning of differential calculus, as mandated by the National Curriculum Statement for Mathematics, grades 10-12 (DoE, 2003). The focus was on the use of the zone of proximal development (ZPD) to learn differential calculus. Vygotsky (1978) defines ZPD as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers"(p. 86).

Borchlet (2007) asserts that "learning is determined by the interactions among the students' existing knowledge, established social context, and the problem to be solved" (p. 2). This supports Vygotsky's (1978) idea that higher order thinking develops first in action and then in thought. Borchlet (2007, p.2) argues that "the potential for cognitive development is optimized within a zone of proximal development or an area of exploration for which a student is cognitively prepared, but requires assistance through social interaction".

Lerman (1996, p.4) defines approaches which are informed by socio-cultural theory to Mathematics teaching and learning as involving “frameworks, which build on the notion that the individual’s cognition originates in social interactions”. This research adapted the first two in the three general themes of Vygotsky’s socio-cultural theoretical perspective, as explained by Wertsch (1990).

The first theme, namely ‘the use of a genetic or developmental method’ asserts that understanding in the learning process depends on the students’ intellectual capability to assimilate new knowledge. However, intellectual ability cannot work alone; it should be a socio-genetic process with learning occurring through social interactions between students and a more knowledgeable person such as another student or a lecturer (Vygotsky, 1978).

According to the second theme, higher mental functions have their origin in human interactions and appear gradually during the process of radical transformation of the lower functions, while the transformation is made through ‘mediated activity’ and ‘psychological tools’ (Kozulin, 1990; Newman & Holzman, 1993). In this study the students built knowledge of differential calculus from their prior knowledge. The students brought their previous experiences to bear on the new situations that they encountered.

This approach is reinforced by Wertsch who asserts that:

> Any function in the student’s cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an inter-psychological category and then within the student as an intra-psychological category.
>
> (Wertsch, 1985, pp.60-61)

In structural discussion of learning activities, inexperienced students use their prior knowledge while working with more experienced students to gain understanding in the learning of derivatives. As time goes on, the inexperienced students take on increasing responsibility for their own learning and participation in joint activity (John-Steiner & Mahn, 1996). It has been shown that student involvement in academic work improves knowledge acquisition, as well as general cognitive development (Smith, Desimone, Zeidner, Dunn, Bhatt & Rumyantseva, 2007).

Steele (2001) claims that a knowledgeable person can help to add meaning to what is familiar to the student when he or she enters the student’s ZPD. As students continue, they can perform activities independently, which they were only able to perform previously with assistance. This shift in the students’ level of understanding helps them to find a way of attempting problems that they were unable to solve before even with assistance. The learning activities shift from simple problems to advanced problems of each section of differential calculus.

The students can solve difficult problems as they interact with other students, showing and explaining to them as the level of complexity increases in differential calculus problems. Vygotsky (1978) states that "what is in the zone of proximal development today will be the actual developmental level tomorrow, that is, what a student can do with assistance today he or she will be able to do alone tomorrow" (p. 87). As the students engage with structural discussions of learning activities regarding differential calculus they are later able to solve more challenging problems than before on their own.

The interpretation of Vygotsky's socio-cultural learning theory to cognitive development is that one should understand the two main principles of Vygotsky's works namely the zone of proximal development (ZPD) and the more knowledgeable other (MKO) (Galloway, 2001). The MKO refers to someone who has a better understanding or a higher ability level than the student, with respect to a particular task, process or concept (Galloway, 2001). Steele (2001) states that by carefully involving students in appropriate learning activities, the lecturer can create the ZPD so that every student may develop an understanding of mathematical concepts that have been established through culture.

METHODOLOGY

This study was located within an interpretive paradigm that aimed to analyse students' experiences in the process of learning differential calculus. The researcher employed a qualitative research for this study.

This research took place at a university of technology in Cape Town, South Africa. Sampling was purposeful and the selection was based on students' poor performance background in Mathematics. The sample group was selected from first-year students who registered for Chemical Engineering in the extended curriculum programme in 2008. The researcher focused on a single group of 20 Mathematics students in the field of engineering in the second semester of their first year academic year.

The researcher collected data through students' written work, which entailed class exercises, tutorial tests, sample tests and formative and summative assessment tasks, observations, audio-visual recordings and in-depth interviews.

Audio-visual recordings were used to obtain accurate observations. The twenty students were arranged into five groups of four students in each group. They worked together and elected one member of the group to report their calculations to the whole class. In their reports, the researcher and other students intervened by questioning the reporter and group members for further clarity and improved understanding.

The researcher used audio-visual recordings to observe students completing calculations on the board, while revising a sample test as preparation for the final examinations. In the in-depth interviews four main questions were asked with some probing questions, depending on the response of the interviewee. The following questions were asked:

- What did you learn in differential calculus in high school?
- What benefit, if any, did you find in the use of additional learning materials during classroom interaction?
- How do you perceive the use of structural discussion of learning activities to learn differential calculus?
- Which section of differential calculus did you find difficult?

This article reports findings based on observation, audio-visual recordings and in-depth interviews.

For analysis purposes the data was systematically and thoroughly organised, analysed, classified, and grouped into patterns and themes. Themes began to emerge from the data about what happened when the lecturer helped students to learn differential calculus through communication. In terms of socio-cultural perspectives, students share their reasoning about ideas with others and listen to others as they share their thinking.

FINDINGS WITH RESPECT TO STUDENTS' RESPONSES IN AUDIO-VISUAL RECORDINGS AND OBSERVATIONS

Through audio-visual recordings and observations the researcher observed six students completing calculations. Each student represented a particular group. The student from the first group calculated the derivative of $y = x \quad - \quad {}^{x} + e$. This student demonstrated understanding through following appropriate procedures and reached the solution and answer correctly. The calculation was as follows:

$$y = x \quad - \quad {}^{x} + e$$

$$\frac{dy}{dx} = \; x^{\;\;-1} - \; {}^{x} \ln \quad + 0$$

One student suggested a move of further differentiating $\ln$ .This student showed that he took as a variable, not as a constant, and he did not understand when differentiation is over. This gave the researcher an opportunity to explain and e as constants with special values such as: $\cong 3{,}14$ as an irrational number and $e \cong 2{,}7182818284$. Many students enter university first-year level in Mathematics with an assumption that numbers are only written in Arabic symbols. Hence they do not treat or understand numbers such as and e as constants.

The student from the second group calculated $y = x^3 \tan(x^3 + 2)$. In the calculation the student demonstrated understanding of the problem, but also demonstrated poor understanding of a dot as a multiplication sign, as she used a dot and brackets simultaneously.

The student from the third group calculated the derivative of $y = \frac{e^{2x}}{2} + \frac{2}{e^{3x}} + \frac{x}{e^3} + \frac{e}{x}$. This student shifted the denominators up to make negative indices and she applied the power rule to differentiate.

She changed the original problem to $y = \frac{1}{2}e^{2x} + 2(e^{3x})^{-1} + x \cdot (e^3)^{-1} + e^3(x)^{-1}$. In the second step she tried to differentiate to obtain $\frac{dy}{dx} = \frac{1}{2}e^{2x} \cdot 2 - 2(e^{3x})^{-2} \cdot 3 - xe^3 - e^3x^{-2}$.

In this question the student showed that she did not understand that she cannot apply the power rule to differentiate e^{3x}. At the same time the student did not treat e^3 as a constant, but instead treated e as a variable.

The student from the fourth group calculated $y = \cos ec4x + \ln(\cos x)$. This student demonstrated the understanding of this problem, as it requires understanding of the derivative of cosecant function, natural logarithm of a cosine function and the chain rule. One student asked: “How do we see that a problem requires application of chain rule?” The researcher referred the question to the class. Another student’s response was that we use the chain rule when we cannot apply other rules. No student in the class could provide a satisfactory answer for this question. The researcher referred the students to the learning material which described chain rule. The researcher also explained chain rule as a composite function rule by using the following four examples to illustrate that:

1. $y = x$;
2. $y = (x+2)^3$;
3. $y = (3x^4 + 1)^3$; and
4. $y = \cos ec4x$

Another student asked the difference between the product rule and the sum rule. The researcher’s response was to find out from the same student the meaning of a sum and a product. The student’s response showed that she knew sum and product as the answer of addition and multiplication, respectively. The researcher explained the two rules, namely the sum and difference rule, and the product rule by means of examples. The first example was to explain the derivative of $x^2 + 3x + 1$.

The second example was to explain the derivative of $3x^2 \tan x$. This was differentiated by treating $3x^2$ as the first function and by treating $\tan x$ as the second function.

The process of observation assisted the students to raise their questions in the learning of differential calculus with respect to interpretation of the chain rule. The students also had an opportunity to pose questions about differentiation rules in order to improve their understanding.

The student from the fifth group calculated $y = \cos^2 4x^2$. This student transcribed the problem incorrectly. Instead of writing $y = \cos^2 4x^2$, he wrote $y = \cos^2 (4x)^2$. The student was also confused by $y = \cos^2 4x^2 = (\cos 4x^2)^2$. None of the students noticed the mistake at the beginning. They only realised that there was a mistake when they noticed that they had different answers and then tried to find the error. Subsequently, one student from the audience noticed that the nature of the problem had changed. This supports the view that within the classroom, the person who is more knowledgeable is not always the lecturer; students can also be placed in collaborative groups with others who have demonstrated mastery of tasks and concepts (Coffey, 2009). Another student who is regarded as an expert in tasks of this nature solved the problem correctly, thus showing an understanding of procedures.

The student from the sixth group calculated $\frac{\cos x}{1+\sin x}$. This student showed an understanding of the quotient rule and the derivatives of cosine and sine functions. The calculation was as follows:

$$= \frac{-\sin x(1+\sin x) - \cos x(0+\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

The minus sign was taken out as a common factor to establish the identity $\sin^2 x + \cos^2 x = 1$.

In $\frac{-\sin x - 1}{(1+\sin x)^2}$

$$= \frac{-(\sin x + 1)}{(1+\sin x)^2}$$

The minus sign was taken out as a common factor to establish like terms in the numerator and denominator in order to obtain

$$= \frac{-1}{(1+\sin x)^2}.$$

The calculation demonstrated an understanding of trigonometry identities, as well as an ability to simplify. The questions from one student showed a poor understanding of basic algebra and a poor understanding of trigonometry identities. The said student did not understand the reason for substituting $\sin^2 x + \cos^2 x$ with 1, and did not understand the reason for taking -1 as a common factor.

FINDINGS WITH RESPECT TO STUDENTS' COMMENTS DURING IN-DEPTH INTERVIEWS

The responses of the students in the sample of the interview showed that many students had studied the first principles of differentiation in their high school curriculum. One of six student interviewees claimed that they did not study any portion of differential calculus in their high school curriculum. Four of six student interviewees stated that they had studied the first principles of differentiation and the power rule, although it was not named as the power rule. Only one of six student interviewees asserted that in their school they had studied first principles, the power rule, graphs, and the application of differentiation. Five students stated that they had not worked with fractions in their studies of the first principles of differentiation.

The students' responses showed that although they had learnt the first principles in high school Mathematics, it was taught superficially. The learning of the limit definition of a derivative should focus on assisting students to develop differentiation rules. The students entered tertiary level without understanding application of the first principles to develop differentiation rules. One may argue that high school Mathematics teachers probably do not use the limit definition of a derivative to develop differentiation rules.

The students' perceptions indicated that the use of structural discussion of learning activities had assisted them to develop regular practice to learn differential calculus. They also learnt to practise in groups and learnt from their mistakes and the mistakes of other students.

A student's remark in a differential calculus lecture where one student claimed that the fact that they are given a table of derivatives in examinations means that there is no point in learning the proof of the basic derivatives such as $\frac{d(\cot x)}{dx} = -\cos ec^2 x.$ This remark shows that some students tend to assume that it is sufficient to know the derivative of any trigonometry function without having to prove it. The students tend to rote learn the derivatives as they appear in a table of derivatives and, as a result, they do not understand how mathematicians develop differentiation rules and some of the basic derivatives. This study showed that students enter universities with instrumental understanding as a key way of learning Mathematics. The structural discussion of learning activities should lead to meaningful learning of differential calculus in which students are able to understand the links and relationships that give Mathematics its structure and, which aids motivation and is beneficial in the long term (Skemp, 1976; 1977). The fact that they are satisfied to be given a table of derivatives without showing the proof shows that they are not familiar with tracing the origin of differentiation rules.

The students' responses showed that collaboration in the learning of differential calculus kept them updated: they practised regularly and it also built confidence in their learning process, as they received immediate feedback in the process of learning differential calculus. They also highlighted that an advantage of the use of structural discussion of learning activities was to practise examples where students were able to consult the solutions when they were stuck.

The students stated that the use of learning activities alone was not enough for them to master the content of differential calculus what is covered in their extended curriculum programme. They explained that their understanding had developed through classroom discussions with other students and interaction with the lecturer.

CONCLUSION

The students' responses in the audio-visual recordings and observations showed that the use of social interaction in discussions and negotiations assisted them to gain confidence to learn differential calculus. Through challenging their calculations and procedures during classroom interactions, the students had an opportunity to question one another to obtain clarity in their learning process. The open discussions during classroom interactions also made it easier for the students to pose questions to the researcher to explain the use of differentiation rules in the learning of differential calculus. The students' calculations showed an improvement with regard to their understanding of differential calculus within the scope of the extended curriculum programme.

The interview responses confirmed that the use of structural discussion of learning activities assisted the students in the sample to gain confidence when learning differential calculus. The students in the sample also confirmed that they had not received enough knowledge during the lecture periods and in classroom interaction to master the derivatives of trigonometry reciprocals and transcendental functions such as x^x. They also revealed that they had learnt little about differential calculus at high school.

In examining the students' understanding of differential calculus it became clear that there was a gap between their high school learning and university first-year level. It appears that university lecturers assume that students who have matriculated are able to derive differentiation rules through engagement with the limit definition of a derivative. The students in the sample group showed that although they had learnt the first principles, they did not work with the first principles to trace the origin of differentiation rules. The findings of the study suggest that the researcher should revise the limits with students in the sample group to involve the use of the limit definition to derive the rules of differentiation.

Learning activities should be designed to allow students to explore the understanding of standard derivatives. The students' responses and performance throughout this study demonstrated that little or no attention is given to the development of differentiation rules in high school. Hence the students either rote learn the rules or tend to rely on tables that are given in the examination room. The study also revealed that the learning material, which is used to study differential calculus tends to pay little attention to the reciprocals of trigonometry functions.

Learning activities that are designed should involve a variety of differential calculus problems, which focus on the derivatives of trigonometry reciprocals. The students' performance also showed that extra time should be devoted to the unpacking of the chain rule to solve differential calculus problems. Students should also form study groups to facilitate their learning process throughout the year of their extended curriculum programme. The use of ZPD as a framework, in which a teaching approach can be developed towards the enhancement of students' understanding, seems to help them to gain an understanding of differential calculus.

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NOT ADDING UP: REFLECTIONS ON (NOT) LEARNING MATHEMATICS AND COLLECTIVE MEMORY WORK

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This paper discusses a portion of a larger research project in which I was trying to gain a deeper understanding of students who chronically failed mathematics. The young women, mostly seniors in a New York City public high school, were in danger of not earning their high school diplomas because they could not pass the New York State Regents examination in mathematics. I wish to discuss my findings from a methodology I used called "collective memory work" and share some of the insights I learned about the students and their perceptions of themselves as learners of mathematics. Readers should be able to reflect upon their own practice and have greater insight into how to reach students who are experiencing failure in mathematics.

INTRODUCTION

The main research question that drives this work is:

How is subjectification in mathematics for young women of color from under-funded communities accomplished?

a. How have the discourse(s) of mathematics acted on the young women in this study and;

b. how have the students responded?

The research methods selected for this study are uniquely suited for addressing these questions The research method is a hybrid, combining collective memory work, observations, and action research through engagement with students in doing mathematics. Collective memory work is designed to reveal "the social and discursive processes through which we become individuals" (Davies, 1994, p. 83). Through the process of collectively telling their stories/memories of learning school mathematics, the participants should begin to see what mathematics discourses (the "social or discursive processes") have been influencing them.

Memory work is a research strategy initially created by Figga Haug and her colleagues (1983). Memory work helps participants to review past experiences and come to understand how society works to construct them while simultaneously revealing how they actively participate in these constructions (pp. 34-35). Initially, participants in collective memory work decide what topic they want to talk and write about. Given the purpose of my research, I told the students they would be sharing stories of their first memories of learning mathematics. As participants tell a story, listeners must try to visualize themselves in the story. If they cannot do so, they ask the teller specific questions, such as, “Could you feel your heart beating?” to get at the details that make the story more embodied. The telling of stories usually trigger additional memories of other stories that can be told to the group. (See, for instance, Davies, 2001). Haug et al. (1983, p.47) explain that:

> once we have begun to rediscover a given situation-- its smells, sounds, emotions, thoughts, attitudes — the situation itself draws us back into the past; ... it allows us to become once again the child — a stranger — whom we once were. With some astonishment, we find ourselves discerning linkages never perceived before: forgotten traces, abandoned intentions, lost desires and so on.

One very important aspect of collective memory work is that the stories are told in a group. The *collective* nature of storytelling allows for the shift from stories that state “this is my identity; to stories which illustrate: this is how we have collectively experienced [the story event, such as] silence — this is how silence is done by us and by others, and with these powerful effects” (Davies, 1997, p. 64). This shift to collective experiences also reveals how subjectification is accomplished. As Davies (2000b, p. 169) explains,

> We are subjected through discourses and within relations of power, and there is no clear boundary between what we are or are in process of becoming and those discourses through which we are subjected. But...discourses/texts/thought are not static...In the processes of becoming speaking, knowing subjects, we become subjects in transition, subjects who can use the powers that their subjectification by and through discourses gives them, to trouble, to transform, to realign the very forces that shape us.

Thus, one possible outcome of collective memory work is this ability to “trouble and transform” mathematics discourses. However, school mathematics presently requires definite knowledge (<u>**one**</u> mathematics discourse) and simply being aware that it is <u>**a**</u> discourse allows one to see how the discourse might be changed.

Scott (1991, pp. 779-780) further explains,

> Making visible the experiences of a different group exposes the existence of repressive mechanisms, but not their inner workings or logics; we know that difference exists, but we don’t understand it as relationally constituted. For that we need to attend to the historical processes that, through discourse, position subjects and produce their experiences. It is not individuals who have experience, but subjects who are constituted through experience. Experience in this definition then becomes not the origin of our explanations, not the

> authoritative (because seen or felt) evidence that grounds what is known, but rather that which we seek to explain, that about which knowledge is produced.

The purpose of this work was to gain a deeper understanding of how some young women of color have been subjectified by school mathematics. Their remembered experiences of learning mathematics in school become a window through which we can begin to understand the mathematics discourses present in schools and in society. I understand their memories of mathematics as processes by which they were constituted as "students of mathematics" and as "failing (or disengaged) students of mathematics." This work seeks to explain how their experience was produced even in a girls' academy where teachers were intent on helping them become successful students of mathematics.

I wish to briefly explain the framework of this research project. I met with eight young women of color who were unable to pass the New York State Regents examination in mathematics (referred to as a Regents). It was an all female school with students from grades seven though twelve (ages twelve to eighteen). Students usually take the mathematics Regents at the end of their freshman (first) year of high school. Most of the students in the study were entering their senior (fourth) year and had yet to pass the examination. A special class was set up for them as the school personnel were determined to help them graduate. I observed the participants once a week in a morning review session. My purpose during these sessions was to watch and record how the students actually completed mathematics problems. The data from these observations are beyond the scope of this paper. I also met with the students one afternoon a week for the entire ten-month school year. We were scheduled for a class period during school hours after their special mathematics class. During these sessions they wrote and told stories using the methodology of Davies, et.al. I tape recorded and transcribed these sessions.

During the spring semester, the participants in the study actively avoided the special mathematics class set up for them. The students learned how to "work the system" established by the New York City Board of Education. While each teacher took and submitted attendance at the start of every class, parents were only notified if the students did not attend their second period class. Official attendance records were based upon being present during the morning "homeroom" session. The students in this study began to hide in stairwells, or bathrooms, or empty classrooms during mathematics. The teacher, who already had a full load of classes when she agreed to work with these girls, was unaware of how often they were skipping class. I was able to get their attendance records, and each participant was attending less than half of the mathematics class by mid-April.

I am going to focus on only one of the young women's stories in order to demonstrate the possibilities of collective memory work as a means of gathering data. I will then discuss some of the findings from across the stories. Nadine is African American, and was an 18-year-old senior at the time of the study. She had met all of her Regents requirements (i.e. English, history, science, and foreign language) except for mathematics. She had taken the Mathematics A (integrated algebra) class three times with three different teachers. She had taken and failed the Mathematics A Regents test five times. Nadine wrote the following story:

*I attended a public school on the lower East side of Manhattan. African Americans were very scarce there, not because of Caucasians but because of other people of color. It was a very multicultural school, mostly Hindu, Spanish, Chinese and for some reason the Hindu dominated science and the Chinese dominated art and math. They made me feel real dumb, (*really stupid, in math class*). It was in Mrs. R.'s class* [third grade, age 9]*. She gave the class a long division test after only two days since she had taught long division to the class. Taking a single number and dividing it into a larger number really frustrated me. I saw no point if I had a calculator, (*and I've had a calculator since, like, second grade*). So, it was January 30th, after --two days after my birthday, after I blew out the candles on my cake when I was at home, I received my report card. Red U* [unsatisfactory] *for math. Damn, I hate math* [4/1/03].

The non-italicized words in parenthesis were not written in Nadine's story, but when she read the story aloud with the group, she said them. After she finished reading from the text she had written, she said,

> My mom had said, 'Here, you got this from school.' She thought it was a paper, but it was my report card. I got a red U, which is unsatisfactory--the worst kind of unsatisfactory you can get. So, that was my birthday present and ever since then I hated math. Since third grade--since my birthday in third grade-- I hated math. And that's it…It's the truth. I knew I was really, really, really bad at it. I think it was 'cause I saw everyone else getting it so easily and that frustrated me even more. It made me think that maybe it wasn't meant for me to do it. And maybe in third grade, feeling really stupid and thinking that maybe it was just for Chinese. I just wasn't supposed to do it. I started ignoring it, and I think that's why I am so bad at math now because if I don't understand it, I ignore it. I think, I'll get it later, but I don't really do it. And maybe that's why I don't know what I should know about math [4/1/03 p. 3].

Nadine describes a shift I observed in all the participants. They started to describe themselves as **unable** to do mathematics, rather than as finding it difficult or as not understanding it. Her written story describes "those others" who are able to do mathematics. In her experience, those "others" were the Chinese students. She also mentions that her teacher gave a test after only teaching the material for two days. When she spoke about the story, she described how she hated mathematics. Then, she made the shift– in the paragraph above she said that mathematics made her feel stupid, and that she was "not supposed to do it" and she "ignored it." Nadine had inscribed herself as one who is unable to do mathematics. It seemed that Nadine had at least two other available discourses. She could have blamed her teacher (because, according to her memory, the test happened too soon) or could have suggested that the Chinese students were favored. Instead, Nadine internalized a perceived lack about herself. Mathematics became something that was not meant for her. Her statement, that she ignores mathematics and puts it off until later, was exactly how she dealt with mathematics as a high school senior.

When I asked Nadine to elaborate on her feelings about the red U, she said,

> It really sucked, especially because it was my birthday, and it said I didn't try hard enough and that sucked even more because I did try and I just didn't get it. I'm like retarded in math…I've admitted it's not for me. But as long as I don't suck at like every other subject, it's OK. I can accept that as it is [4/1/03 p. 4].

Here, Nadine says that mathematics is "not for her" and in other interviews, she said, "It's not my thing," "I'm not feeling it," or simply, "I can't do it." It is interesting that Nadine blamed the teacher for giving a test after teaching division for only two days, but she still internalized the teacher's red U.

Often, after telling stories in collective memory work, participants then write poems. The poems are allowed to have literary devices and metaphors that the stories are not. (See, for example, Richardson, 1997 or Gannon, 2001).

Chris is a Puerto Rican-American and was a sophomore during my data collection. She had taken the Mathematics A course for a year and a half, but her teacher did not think she was prepared to pass the examination. She joined the afternoon class in January. Chris wrote this poem after we had discussed her story at length. The words seemed to flow out of her and she barely paused as she wrote:

Remembering the day

Remembering the day when math was difficult

Remembering the day when I just didn't understand

It was a clear sunny spring day,

and all I remember was I wanted to play,

outside with everyone else.

Didn't feel like being trapped up in a hot room.

Opening books and writing equations

Made me feel like I didn't have my child like connections
Sitting in class listening to the teacher teach,
as if he were a preacher, preaching a preach.
Looking over x and y graphs
Made me feel like I was a puzzle piece
with all this burden all this grief
on my mind,
Just because I don't get the equation
I've completely lost my math connection
Asking for help, so I can go on into the math world,
Wanting, needing, begging for help so
I can get this burden off my soul
because it's placing a hole in my mind.
Never getting help, never succeeding in
understanding graphs.
I remember the day, remember the day
when graphs were a blur . . . . . . [4/15/03]

The young women in this study positioned themselves as "not able" to do mathematics. They used this "unable" category as a sorting tool, dividing people into groups of those who were able to do mathematics (those "other" people) and those who were unable to do mathematics. This construction of "able" or "not able" to do mathematics remained unchallenged by the students and their teachers. School mathematics, within the context of city and state tests, reinforced this construction. One afternoon, three of the students were talking about how they started to accept the fact that they would not get good grades in mathematics.

Jaz	But you can't cheat on word problems. But there was a girl next to me and I would look at her answers quick. I'd still be stuck. Her name was Darlene.
Nadine	Something about math didn't work for me...(it) was like you start getting grades that are like 65 and I start being all happy. [65% is the minimum passing score.]
Jaz	Then you just keep on aiming for a 65.
Nadine	Exactly. And you lower all your standards until you think that's all you need and you accepted it. Then it turned into my mother getting happy with a 65!

Jan	Not my mother--she's not happy with a 65. I need to be up there in the 70s.
Nadine	But my mother knew that math was not my thing...
Jan	(Interrupted) When you keep on saying that math is not your thing it's like you don't have control. If you hate something and you keep on saying you hate it, what's that called?
Nadine	(Interrupted) The only reason why you have trouble with math now is because you've accepted it. That's true for all of us [5/3/03]

The students' shift into believing that they were unable to do mathematics fell within broader mathematical discourses. For example, educational psychologists studying failure might frame Nadine's subjectification within the language of attributional style. In a number of studies reported by Licht and Dweck, (1987) girls are more likely to attribute their failure to ability. Ability is described as an attribute that students considered stable, meaning that they conceptualized that there is a fixed amount of it, and, therefore, beyond the their control. The authors found that students who attribute their failure to something that is stable tend to lower their expectations for future success and are less likely to increase their efforts. Students who attribute failure to that which is unstable, usually a lack of effort, are more likely to have high expectations of future success. Nadine's story echoes their findings as she kept distancing herself from mathematics and showing less effort because she believed it was "not her thing."

Policy makers tend to use the language of effort, as do Licht and Dweck (1987). The underlying belief is that if students and teachers would simply work harder, the students would be more successful. The students in my study believed that they were making every effort to be successful in mathematics. Nadine said that part of the reason why the big red U "sucked" was because it suggested that she was not trying hard enough. Nadine resisted the discourse that suggested that passing was all about effort. She believed that she did not have the ability to learn mathematics, and that had become part of her identity.

There is a societal discourse that suggests that mathematics is for some and not for others. Robert Moses (2001) notes that there is a cultural acceptance of "illiteracy" in mathematics that does not exist in other subject areas. It is perfectly normal to hear well-educated people say, "I am just not good in math." In addition, parents do not appear as concerned about their child's failure in mathematics as they are in other subjects. The students recognized this when they spoke of their mothers being happy with a 65 or somewhere in the 70s. While Moses (2001) does not specifically talk about race and class in his discussion of parents and mathematics, other authors do. For example, Drummond and Stipeck (2004) write about low-income parents of third and fourth graders and their beliefs about their role in children's learning. They found that the parents of third graders rated the importance of helping their child with reading as more important than helping them with mathematics. The authors also found that more than half of the parents responded that they felt inadequately prepared to assist their child in mathematics, while all of the participants felt that they could help their child with reading.

Other researchers have looked at the relationship between parents' gendered stereotypes and students' beliefs. Tiedemann (2000, p. 149) found that "[p]arents who think in stereotypes assign lower ability in math (but not more effort) to their daughters than to their sons." The author also found that the parents' perception influenced the child's self-perception of themselves as learners of mathematics.

Nadine said that her mother understood why she was not good at mathematics, because her mother was not good at it either. When Nadine shared that, Jaz and Chris agreed. They said that their low mathematics scores did not seem as important to their mother's as other low scores might have been. These parents support the cultural notion that mathematics is not intended for everyone. Moses (2001, p. 9-10) writes, "math instruction weeds out people and you wind up with what amounts to a priesthood, masters of the arcane secrets of math through what appears to be some God-given talent or magic."

Nadine's words mimicked the language of the studies reviewed above. She said that the red U "sucked," in particular, because the teacher said she did not try hard enough. Yet, Nadine believed that she had tried, and was left, then, to blame her failure on her lack of ability. Licht and Dweck (1987, p. 96) call these students "learned helpless," because, "their causal attributions imply that the termination of failure is beyond their control." They suggest that explicitly teaching children that their failures are a result of lack of effort should help students perform better. This suggestion seems to contradict Nadine's experience and seems too simple a solution for such a complex issue of internalizing failure. The authors' solution does not account for the multiple factors that may contribute to failure, such as students' past achievement, social environment, or community context. However, it seems as though Nadine had, in fact, come to think of her failure as something completely beyond her control. She believed that this failure in mathematics was acceptable, too, because she did better in other subjects.

Throughout this study, the students appeared to be creating stories that allowed the categories of "able" and "unable" to remain stable. Davies (1989), in explaining the complex weavings of identity, writes that there is "the emotional meaning attached to each of those categories which have developed as a result of personal experiences of being located as a member of each category, or of relating to someone in that category" (Davies, 1989, p. 230). The students in this study had a strong emotional need to distance themselves from their own inability. The participants stopped attending mathematics class regularly despite the possible high-stakes consequence of not graduating. When the students would return to class, after having missed instruction, they felt even more unable, because they did not know what was happening in the class. This pattern then had to repeat itself-- not coming to class, feeling unable upon returning, and thus leaving for even greater periods of time.

Haug and her colleagues (1987, p. 150) explain at length:

> We have a spontaneous desire to be confirmed in our thoughts and actions, in the intuitive judgments we have grown used to making. In empathizing with and spontaneously affirming a number of the stories told by the group, we not only reinforced each other in our mutual experiences, but also in the preconceptions and simplistic explanatory models they conjure up in our minds. In doing so, we consolidated old assumptions, and made it impossible to question these assumptions in order to gain new knowledge of the issues.

Jan, in the conversation about lowering standards, attempted to disrupt the simple explanatory model from which we were working. She said, "When you keep on saying that math is not your thing it's like you don't have control." Nadine's response that they have trouble because they have all accepted their failure indicates that she believed the participants had some responsibility for their struggles. But they were interrupted and that thread of the conversation was never picked up again. I was actually surprised when I listened through the hours of tapes. All of us failed to pay attention to the alternative discourse, and our silence suggests how we reinforced the model.

At the conclusion of this study, six of the eight participants took the Mathematics A Regents examination in June, but none of them passed it. One of the seniors decided she did not want to graduate after she became engaged to her boyfriend. The other senior was evaluated for a learning disability and was able to take a mathematics competency test instead of the Regents. A few days after the examination was given, the State of New York declared the results of the Mathematics Regents invalid (see Goodnaugh, (2003) and Herszenhorn, (2003). The State's decision allowed the four seniors who took the exam to graduate without mastery of the mathematics content. The two sophomores had to attend summer school and take the Regents again.

The work of Davies and her colleagues has helped me understand the potential for poststructuralist theories as applied to research methodology and collective memory work. The students' earliest memories of learning mathematics, and, in the case of these participants, feeling as if they failed mathematics, all occurred in third or fourth grade. Yet, the students in this study never seemed able to find an alternative discourse to the ones they developed early in life. Part of the major frustration in trying to complete this work was the students stopped attending class on a regular basis. The school system, the Women's Academy, and the regular classroom teacher were all complicit in their absences. I, too, felt as if I had failed the students by not creating enough of a bond with them that they would consistently attend my work sessions with them. It was discouraging to witness their despondence over failing the examination again.

The results of this study offer numerous educational implications. Perhaps most importantly is the notion that girls as young as third or fourth grade are distancing themselves from the field of mathematics. It seems that as they start to create an identity of themselves as unable to learn mathematics, they choose behaviors that reinforce this notion. More research should be done that looks at possible early intervention programs for students who struggle with mathematics in the primary grades. Collective memory work also provides a researcher with powerful insights into students' identity formation, and I believe more such work would be highly beneficial to the mathematics education community. Finally, New York State must continue to examine how they assess the mathematics learning of the students. The Regents examination in mathematics has been fraught with issues over the past decade, and the move toward even more standardized testing is problematic throughout the State.

As a classroom teacher, I have looked for opportunities to hear the students' opinions, thoughts and feelings about mathematics since I completed this study. More importantly, I have tried hard to disrupt the discourses that suggest that it is acceptable to be a poor student in mathematics. I frequently speak to teachers who say, "Oh, I was never good at mathematics" and ask how they would feel if I said, with equal lightheartedness, "Oh, I am illiterate." This simple response helps people understand how far we have to go in promoting numeracy as strongly as we promote literacy. I believe that we must continue to seek ways to allow the students' voices to be heard. We, as researchers and educators have to provide children with places where they can speak and we must take heed.

Looking over x and y graphs

Made me feel like I was a puzzle piece

with all this burden all this grief

on my mind,

Just because I don't get the equation

I've completely lost my math connection

Chris [4/15/03]

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AN INVESTIGATION INTO LEARNERS' APPROACHES TO SOLVING PROBLEMS IN MATHEMATICAL LITERACY

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Mathematical Literacy was introduced as a subject in the South African Curriculum in 2006. In 2008, the first grade 12 external examination was written in the subject. While the grade 12 results in Mathematical Literacy have been encouraging, one area which is a cause for concern is the poor performance of learners in Mathematical Literacy paper 2. This paper attempts to provide some insight into the way learners approach typical Mathematical Literacy paper 2 problems.

INTRODUCTION AND BACKGROUND

Prior to the introduction of the new curriculum (National Curriculum Statement) in South Africa in 2006 for grade 10 learners, approximately 40% of the learners took no mathematics at all in the Further Education and Training (FET) Band. The introduction of Mathematical Literacy into the curriculum made it compulsory for all learners in South Africa to leave school with some form of mathematics. (Perry, 2004)

Mathematical Literacy is defined as follows in the National Curriculum Statement (Grades 10 - 12)

"Mathematical Literacy provides learners with an awareness and understanding of the role that mathematics plays in the modern world. Mathematical Literacy is a subject driven by life-related applications of mathematics. It enables learners to develop the ability and confidence to think numerically and spatially in order to interpret and critically analyse everyday situations and to solve problems" (DoE, 2003:9).

This emphasis on problem solving is repeated in the statement that Mathematical Literacy contributes to the achievement of one of the critical outcomes of the new South African curriculum in the following way: *"use mathematical process skills to identify, pose and solve problems creatively and critically"* (DoE, 2003:10)

Problem solving is further emphasised in the scope of the Mathematical Literacy curriculum where one of the statements focuses on real-life problems. This statement, "*Use numbers with understanding to solve real-life problems in different contexts including the social, personal and financial",* clearly epitomises the true nature of Mathematical Literacy as a subject. (DoE, 2003:10). This is in keeping with Sawyer (2005) who says that a person can be classified as being mathematical literate if s/he is able to apply mathematical learning to the solution of real life problems. Sawyer also highlights the view by authors such as D'Ambrosio, Lambdin and the NCTM that problem solving is central to mathematical literacy at all ages (D'Ambrosio, 2003; Lambdin, 2003 and NCTM, 2000 in Sawyer (2005)).

In the South African context, it would seem that that problem solving is not only an integral part of Mathematics, but also an integral part of Mathematical Literacy. In this regard, there should be various classroom activities which promote and develop the problem solving abilities of Mathematical Literacy learners. This aspect is clearly emphasised in the assessment of the subject. The final examination consists of two papers. Paper 1 focuses on basic knowing and routine applications. Questions in this paper are set at the "*knowing*" level (level 1) and "*applying routine procedures in familiar contexts*" (level 2). Paper 2 focuses on applications, reasoning and reflecting and is a "problem solving" type paper, Questions in this paper are set at the "*applying routine procedures in familiar contexts*" (level 2); "*applying multi-step procedures in a variety of contexts*" (level 3) and "*reasoning and reflecting*" (level 4). (DoE, 2008)

Despite the various curriculum documents suggesting that problem solving is a key area of Mathematical Literacy, a study by Venkat, Graven, Lampen & Nalube (2009) on Mathematical Literacy taxonomy revealed that problem solving only really comes into play at level 3 of the taxonomy. This indicates Mathematical Literacy Paper 2 (which comprises level 2, 3 and 4 questions) is more complex and requires a higher level of cognition than paper 1. The writer who was a Grade 12 internal moderator of Mathematical Literacy in 2008 and 2009 observed that learners tended to obtain marks that are about 10% lower in Paper 2 when compared to Paper 1. This could be attributed to the lengthy reading and copious calculations that are required in Mathematical Literacy paper 2. Adams (2003:786) states that reading mathematics is a multifaceted task because the reader is "challenged to acquire comprehension and mathematical understanding with fluency and proficiency through the reading of numerals and symbols, in addition to words."

One of the questions in a past Grade 12 Mathematical Literacy Paper 2 examination is indicated below:

"Dina and Mpho both practice speed reading in order to improve the time they spend reading texts.

- *Mpho reads at the rate of two words per second.*
- *Dina found that she can read two pages, with an average of 270 words per page, within three minutes.*

Show by calculation whether Dina reads at a faster rate than Mpho"

(DoE, 2008a:3)

This question was part of question 1. Although this appeared to be an easy enough question, there were many different approaches in obtaining the solution. The following was written in the moderator's report about question 1:

> *"Learners who performed below par had poor reading and interpreting skills. They also had difficulty in comparing the data. More specifically, question 1.4 was poorly answered. The memo had six alternative responses for this question, showing that there were different ways of answering this question. Almost all of these solutions came up during the marking. One of the problems learners had was in choosing which operation (multiplication or division) to use in working out who was the faster reader. In many cases, learners did not multiply 270 by 2 to get 540 words on two pages. This meant that they came to another conclusion (that Mpho was faster*)". (DoE, 2008b:6)

It would appear that learners had difficulty with reading, understanding and interpreting the question. It is with this in mind that this research was undertaken, that is, to investigate how learners approach mathematical literacy problems. When the researcher was approached to do a problem solving session with Grade 12 Mathematical Literacy learners, it afforded him the opportunity to conduct this investigation.

SAMPLE

The school was located in a Black township in Port Elizabeth, where Xhosa is the dominant local language. The learners in the sample were selected by the Mathematical Literacy teacher. There were 12 learners in the sample and they were chosen on the basis of two factors; (1) consistent performance in grade 10 and 11 and (2) eagerness and commitment to participate.

RESEARCH QUESTION

The research question for this investigation was "How do learners work through Mathematical Literacy problems?" The following sub-questions were developed in an attempt to answer the research question.

- What knowledge and skills are needed when solving Mathematical Literacy problems?
- Are learners equipped with such knowledge and skills?
- How can learners be supported to acquire these knowledge and skills?

RESEARCH STRUCTURE AND METHODOLOGY

The key focus of this research was on how the learners were able to approach and solve Mathematical Literacy problems. Thus, the experiences of the learners and how they made sense of the problems in their own school environment were the objects of this research, which is qualitative in nature (Hatch, 2002). In this research an attempt is made to describe and interpret how a group of grade 12 learners approach and solve Mathematical Literacy problems, thus locating it in the descriptive research paradigm (Cohen & Manion, 1985). The data was collected from three weekly school visits. The visits were structured, with each visit focussing on specific aspects related to the research. During the **first visit**, the learners of the school assembled in the school library with the researcher and the Mathematical Literacy teacher in attendance. The teacher introduced the researcher to the learners and informed the learners about the purpose of the visit.

The first part of this visit consisted of a survey of both the learners and the teacher using questionnaires. The learner survey was designed to establish how the learners viewed Mathematical Literacy as a subject, find out about topics that were covered in the Mathematical Literacy class and ascertain whether learners were familiar with the knowledge and skills that were required when solving Mathematical Literacy problems. The teacher survey was designed to establish the teacher's views on teaching Mathematical Literacy, the resources used when teaching the subject and how his learners were prepared for the more cognitively demanding level 3 and level 4 questions.

After the completion of the survey, the researcher conducted a problem solving session with the learners. This session consisted of an interactive discussion of the problem solving process in Mathematical Literacy. Details of this discussion appear later in this paper.

During the **second visit**, the researcher briefly revised with the learners what had been discussed in the first visit. After this discussion, learners were given the following problems to work through on their own. These were presented on a sheet of paper which provided space to show their working and calculations, and to provide reasons for what they did. While they were busy, the researcher went around the classroom to have a look at what the learners were doing. The first question was an adaptation of the examination question which is stated earlier in this paper.

> **Question 1**
>
> *Jenny can type on average 65 words per minute. Mary can type two pages (each page has 250 words) in 8 ½ minutes. Who types faster? Show all calculations*

The second question was designed to assess their competency in working with formulae, calculating rates and interpreting answers.

Question 2

Mrs Naidoo decides to buy a frying pan. She has two choices, a round (circular) one and a square one. The round one costs R145 and the square one costs R150

The round one has a radius of 8 cm while the dimensions of square one are 14 cm by 14 cm. The frying pans are placed on a circular stove plate with diameter 20 cm.

2.*1 Which frying pan has the greater surface area in contact with the stove plate?*

Show all calculations (use the formulas below)

Area of a square = side $\times$ *side*

Area of a circle = r^2

2.2 Calculate the cost per cm^2 *for each frying pan.*

2.3 In which frying pan will the heat be more evenly distributed? Give reasons

The **third visit** consisted of two activities, a **focus group discussion** followed by **one-on-one interviews** with selected learners. It is possible that the first two visits would have helped these learners become more experienced and confident about solving Mathematical Literacy problems. In an attempt to consolidate this experience and possible confidence, as well as generate more data, the following issues were addressed during the focus group discussion; *the meaning of literacy when used in* the *context of Mathematical Literacy*; *what they would look for when given a Mathematical Literacy problem to solve and what they felt about the Mathematical*

Literacy problems given to them.

After the completion of the focus group discussion, the researcher conducted interviews with selected learners. This was in connection with their responses to the questions given during the second visit.

THE DATA

The data from all three school visits were analysed with a view to detecting trends and patterns of coherence. During the first school visit, a survey of the learners was conducted via a questionnaire. Some of the key trends and features emerging from the survey with the learners and the teacher are indicated below.

Learners responded very positively when asked about what they liked about Mathematical Literacy. They found Mathematical Literacy to be a very interesting and relevant subject as it gave them an opportunity of **applying mathematics** to their everyday lives. They were able to **understand the work done** in the subject, which was easy to learn, as they did **not go deeper into the calculations**. Formulas were also given and this made answering questions easier, as they did not have to memorise the formulas.

At the same time, learners also indicated some "dislike" about the subject. They complained about the **volume of reading, especially in "word sums",** which had to be done before doing any calculations. They also had to **make notes while reading**. They would rather count and use their calculators straight away. Although the **formulas** given were there to assist them, some found them **confusing**. One learner also mentioned that those who did Mathematics made fun of them.

All learners were very complimentary about what they learnt in the subject. Some mentioned it was an enjoyable subject as "**you learn about everyday things".** Others were more specific by naming content areas such as percentages, simple and compound interest, area and volume and measuring distance. One learner gave an example about knowing about the interest that is charged when buying furniture on hire-purchase.

Learners named a variety of topics, **including some mentioned in the paragraph above, which they found interesting**. These included topics such as simple and compound interest, percentage increase and decrease; calculating income tax; graphs and tables; areas and volumes; mean, median and mode; probability.

Learners were aware that Mathematical Literacy Paper 2 was more cognitively demanding than Paper 1. Thus, for them to be successful in Paper 2, they stated that they needed to **know basic mathematics** and being able to **read each question** carefully and repeatedly until they **understood** the question.

The teacher stated he liked teaching Mathematical Literacy as "it helps develop skills, knowledge, values and other important attributes for the 21st century". At the same time he was **not happy** about the seven assessment tasks which learners were compelled to do as a continuous assessment requirement. His learners did not complete projects and investigations on time, or in some cases, did not do the work, and he found this **very frustrating**. He used **various resources** such as textbooks, worksheets, objects, newspaper articles and pictures which are relevant to the topic(s) being taught. These resources tended to make his **lessons more interesting**. In this regard, he mentioned his lessons on tax calculations, percentages and perimeter and area.

He stated that learners should have basic mathematical skills and know how to use the calculator effectively when answering level 3 and level 4 questions in Mathematical Literacy Paper 2. To help his learners understand these types of questions, he prepared his lessons thoroughly and taught his lessons in an **integrated manner**. He provided an example where he had shown his learners the **connections between fractions, decimals and ratios** and **how to use the calculator** when working with these concepts.

After the administering of the questionnaire, the researcher facilitated the session on problem solving in Mathematical Literacy. This was done in an interactive manner and there appeared to be some synergy between the views discussed in this session and the learners' views in the survey. The following points were discussed in this session:

- The need to read and understand the given problem.
- Highlighting key words and numbers in a problem.
- Looking for tables, graphs and other diagrams and attempting to examine what numerical and other data is conveyed in these representations.
- Making a note of the formula(s) given and using the formula(s) to do certain calculations.
- Understanding the context in which the problem is set and making sense of your answer in relation to the context.
- Checking the units of your answer.

The **second visit** involved learners working through the given set of questions. When the learners were given the questions to work through, the researcher observed that many did not know how to start, possibly through not understanding the language and the context. To get them started, the researcher reminded them about reading through the questions and highlighting key points. They should then see which calculations are appropriate in the given context and identify the operations involved in the calculations. Once learners received this support, they began with their calculations for question 1. For question 2.1, the researcher, once again, observed that learners did not know how to start. They were asked to read the question and examine the formulas that were given. The researcher asked them what should be done when formulas are given. They replied that they should substitute in the formulas, which they then proceeded to do. For question 2.2, researcher had to explain the concept of cost per cm^2, meaning that cost (which was given) had to be divided by area (which they calculated). There appeared to be no difficulty with question 2.3.

To examine how the learners performed in the questions, each learner was given a letter as an identification code and the table below shows how they performed in the questions set. Learners were labelled A, B, C, …, L. The following key was used to determine if learners' answers were correct or not.

Key: Correct √ Incorrect X

Learner	**1**	**2.1**	**2.2**	**2.3**
A	X	X	X	√
B	X	√	√	X
C	X	X	X	√
D	X	√	√	√
E	X	√	√	√
F	X	X	X	√
G	√	X	X	√
H	X	X	X	√
I	√	X	X	X
J	X	X	X	√
K	X	X	X	X
L	X	X	X	X

Table 1 Performance in questions

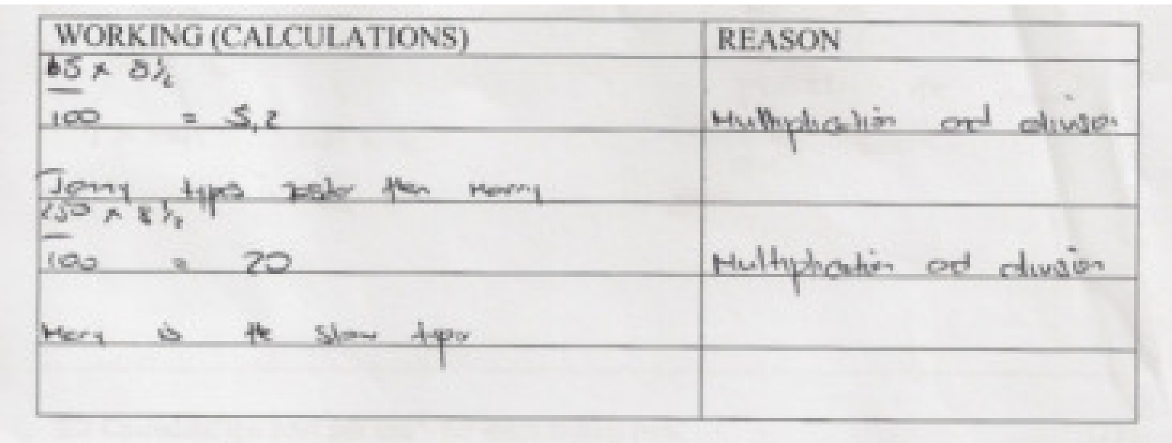
A close scrutiny of the table shows that learners really struggled with the questions. Despite the hints and support given, only two learners were able to do question 1 correctly; they fared no better in question 2.1 which required substitution in formula; the same three learners who had question 2.1 correct, had question 2.2 correct. For question 2.3, most were able to guess that a round or circular frying pan over a circular stove plate is likely to have heat distributed evenly.

The **third visit** comprised a focus group discussion and interviews with selected learners. A summary of the focus group discussion follows.

Learners were aware of the significance of the word “literacy” in Mathematical Literacy. They knew it involved **reading, understanding and interpreting of information** in various forms and recognising what to do when given a question which requires **working with a formula** or **mathematical operations**. They also appeared to know what to do when solving Mathematical Literacy problems. Their responses such as, **identifying if a formula is given, highlighting key information**, especially **numbers** and being able to **think about** how to come up with a possible solution, by looking at what calculations are relevant, were **key indicators** of this ability. However, it would appear that **they were unable** to effectively put these knowledge and skills into practice when answering the questions given. The learners claimed that the **questions were difficult** and they **did not understand the contexts** in which the questions set. They **did not know how to start** and only once they asked the researcher for **clarity and assistance** were they able to understand what to do. They knew they did not fare well in the questions. However, they stated that with **more support and practice** they should do better in future.

The interviews with selected learners were **based on their responses** to the questions. For the sake of brevity and to avoid repetition, only portions of the interviews are indicated. Learners A, B, C and D were interviewed.

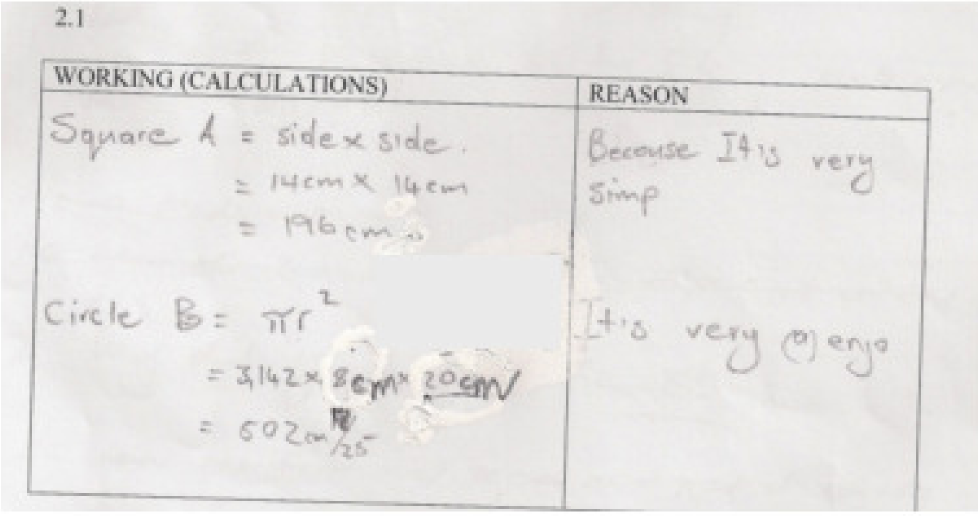
Learner A answered question 1 as follows:



WORKING (CALCULATIONS)	REASON
100 = 5,2	Multiplication and division
Jenny types faster than Henry	
100 = 20	Multiplication and division
Henry is the slow typer	

When the researcher asked learner A why she divided by 100, her response was that this will help her to determine who typed faster. She was also asked what she understood by the 250. She responded that each page consisted of 250 words. However, she conceded that she did not realise that she had to use 500 in her calculations rather than 250.

Learner B's response to question 2.1 was:

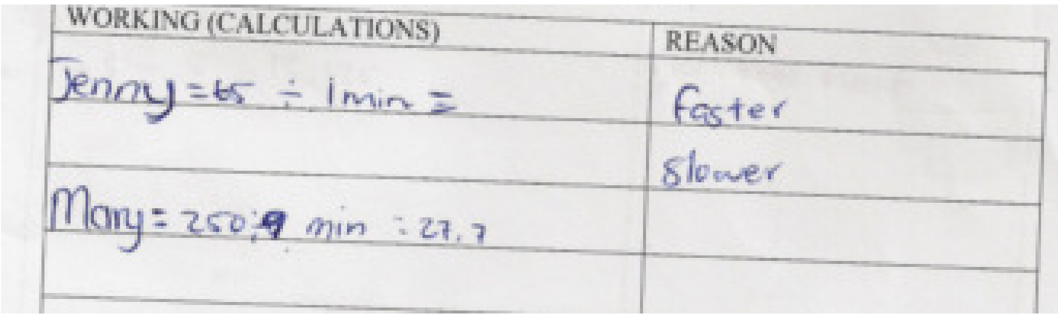


2.1

WORKING (CALCULATIONS)	REASON
Square A = side × side = 14cm × 14cm = 196cm²	Because It's very Simp
Circle B = πr^2 = 3,142 × 8cm × 20cm = 502cm²/25	It's very enjo

The researcher asked learner B why he did the following calculation: $3{,}142 \times 8\ cm \times 20\ cm$. He responded that when he saw the 8 cm and 20 cm, he decided to put it into the area formula. He could not offer any explanation as to why he used two different values for the radius.

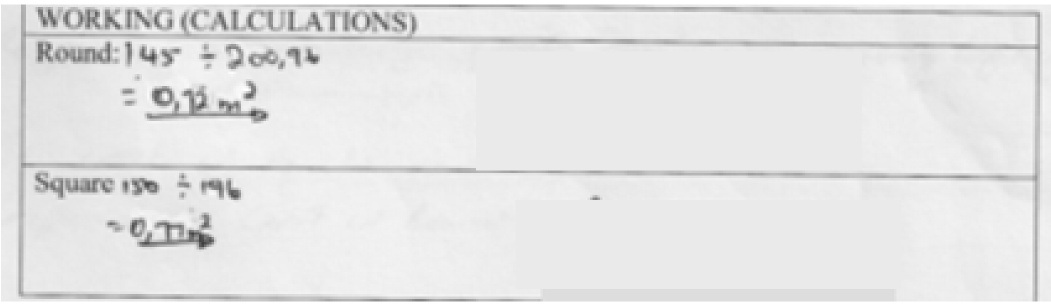
Learner C answered question 1 as follows:



WORKING (CALCULATIONS)	REASON
Jenny = 65 ÷ 1min =	faster
	slower
Mary = 250 ÷ 9 min = 27,7	

The researcher asked her why did she divide 65 by 1 and she responded by saying that she saw 65 and one minute and decided to divide. When asked why she divided 250 by 9, she responded by saying that she thought that she must round off.

Learner D responded to question 2.2 as follows:



She was asked by the researcher what happened to the units in the calculation and where did the m^2 come from. She acknowledged that the units, both R and cm^2 was left out and she made a mistake by writing m^2 instead of cm^2. She was not able to correctly state what the units of this answer should be. The researcher informed her that it should be in Rand per cm^2 or cents per cm^2.

The interviews with selected learners offers an insight into they way they approached the questions. Learners used various methods to solve the given questions, but the calculations **were not meaningful in the contexts** given. It is evident that those with incorrect solutions could not offer any valid explanations for their calculations and solutions. This could be due to their **lack of understanding of the contexts** in both sets of questions (as they claimed during the focus group discussions). As a result, they were **not able to interpret their solutions** in the given contexts. In this regard, they **saw the numbers and formulas in isolation** and not representative of any context. It was also clear from their responses that they **did not attach any relevance or meaning to the units** in the problems, thus, further emphasising their lack of understanding of the contexts.

FINDINGS

The purpose of this research was to investigate the approaches which learners use to solve Mathematical Literacy problems. This investigation involved three visits to a school, using a mixed method approach in data collection. An interrogation of the data reveals the following:

- Learners were very enthusiastic about the subject and were able to mention key topics which were done in class. They were also aware that the subject involved a lot of **reading**, **understanding** and **interpretation** and making **sense of numbers** in a given context. However, they **were not able** to use this **knowledge and skills effectively** during this research.
- Although the teacher claimed that he did cover level 3 and level 4 questions in class, it would appear that this preparation was **inadequate**. When given problems to solve, learners **did not understand** the problems and could not

start. It was only when the researcher gave them support were they able to start.
- Despite the support given, very few learners were able to solve the problems successfully. It was revealed during the focus group discussion that learners **found the problems to be difficult** and they **did not understand the contexts** in which the problems were set. Learners also claimed that they **were not used** to these types of problems.
- During the interviews, learners **were unable to give valid explanations** for their workings. In fact, they were not able to make sense of the context in which the problems were set, nor did they have any idea about **the use of correct units**.

It is clear from these findings that these learners, did not have the necessary knowledge and skills to solve problems which are typically set in Mathematical Literacy paper 2. As shown in this research, they required a great deal of support but receiving this support was no guarantee of success, as was evident from the learners' responses to the problems set.

CONCLUSION

This research was conducted in a Black African township school in an urban area in South Africa and may have relevance for schools with similar demographics. However, it is not the intention to generalise the findings to other schools with, possibly, very different demographics. As stated earlier in this paper, Mathematical Literacy learners write two examination papers, paper 1 which is fairly easy and paper 2, which is cognitively more demanding. While it is possible to pass Mathematical Literacy overall by passing paper 1 and failing paper 2, this approach does not do any justice to the learner as the learner would have probably failed to grasp the key problem solving skills associated with paper 2. It is important that learners are given the skills and knowledge to work through questions in paper 2 with confidence and success. Learners in this research (as indicated in the focus group discussion) believed that they would do better with these types of problems if they were given more support and practice.

Thus, Mathematical Literacy teachers should prepare thoroughly for their lessons by covering a wide range of contexts and mathematical calculations, and ensuring that all levels of questions are addressed. They can also use past year examination papers or develop their own questions and contexts in helping their learners acquire key problem solving knowledge and skills.

Mathematical Literacy is a new subject and much as been achieved in the subject in a short space of time. It is the intention of the researcher that this paper should generate more discussion about problem solving in Mathematical Literacy, especially the role of the teacher in ensuring that learners develop knowledge and skills that would make them better at problem solving.

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ORIENTATIONS TO TEXT AND THE GROUND OF MATHEMATICAL ACTIVITY IN SCHOOLING

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This paper extends previous methodological work of the author—that fashioned a set of categories for describing the grounding of mathematical activity in pedagogic contexts—by relating the categories of ground to: (i) the idea of orientations to the (re)production of texts (Lotman, 1990; Jaffer, 2010), and (ii) the distinction between open and closed pedagogic texts (Eco, 1984; Jaffer, 2011). By organising the categories of ground in terms of Greimas' (1968) treatment of the square of opposition of classical logic and overlaying that with modes of textual orientations and textual openness/closure, taken in pairs, a set of relations between (a) couples of ground categories, (b) modes of orientation to text, and to (c) pedagogic texts as either open or closed, are fashioned. Such a set of relations constitutes a device enabling the generation of descriptions of mathematical activity that are reasonably subtle as well as robust.

INTRODUCTION

In Davis (2010), developing the ideas introduced in Davis & Johnson (2007; 2008), I used the work of Peirce (1931), Hegel (1969), Bernstein (1996) and Badiou (2006) to synthesise a set of four categories for describing the ground of mathematical activity in the pedagogic situations of schooling. Hegel (1969) supplied the proposition that whatever exists has its sufficient ground, and Peirce (1931) showed how to relate ground to that which comes to be expressed (as mathematics in pedagogic situations). As Eco put it regarding the Peircian use of the term 'ground',

> [f]or the moment it is sufficient to recognise that both ground and meaning are of the nature of an idea: signs stand for their objects, "not in all respects, but in reference to a sort of idea, which we have sometimes called the *ground* of the representamen [i.e., signifier]," and 'idea' is not meant in the Platonic sense, but rather "in that sense we say that one man catches another man's idea." (Eco, 1984: 182; italics in the original.)

The important thing to note is that the existential specificities of the object tells us something about the ground that pertains. So, methodologically, determining what kinds of objects are in circulation in pedagogic situations is a crucial element in determining the ground.

The four categories of ground presented in Davis (2010) were referred to as *iconic ground*, *empirical ground*, *propositional ground* and *procedural ground*. The categories were intended to draw out and name the over-determining effects on mathematical activity of the fixing of the primary objects of attention through existential decisions (Badiou, 2006) about what they are *qua* mathematical stuff, and as policed by pedagogic evaluation (Bernstein, 1996; Davis, 2005). The idea of an over-determining effect of an element of a system on the system in which it participates is described by Marx with enviable fecundity in his *Grundrisse*: “It is a general illumination which bathes all the other colours and modifies their particularity. It is a particular ether which determines the specific gravity of every being which has materialised within it” (Marx, 1973: 107). I replace the term ***procedural*** with the term ***algorithmic*** here, so that our four categories of ground are now referred to as: ***iconic***, ***empirical***, ***propositional*** and ***algorithmic***. The categories are described in summary form in Table 1.

Ground	Central grounding resource	Objects of central concern
Iconic	Comparisons centred on iconic features, including similarities and differences of expressions	Graphical and/or symbolic expressions treated as images
Empirical	Empirical testing of expressions	Graphical and/or symbolic expressions treated as in some way “measurable”
Propositional	Knowledge of the mathematical objects and relations between such objects referenced by mathematical statements	Mathematical objects indexed by the axioms, definitions and propositions that are signified by expressions
Algorithmic	Meta-rules governing an algorithm, regulating the selection and sequencing of operations on mathematical signifiers	Operations commonly used within a particular algorithm as well as their sequencing

Table 1: Categories of ground.

I cannot work through all the details supporting the derivation of the categories—which are reported on in Davis & Johnson (2007; 2008) and Davis (2010)—because of limitations on space, so it will have to suffice to note that the categories are concerned with describing the existential specificity of mathematical work as registered in the operational activity of teachers and students, and in terms that emphasise the mathematical resources that have an over-determining, regulative effect on that activity. In enquiring about the grounding of mathematical activity to describe mathematical activity in its existential specificity, Davis & Johnson (2008) take their lead from Hegel (1969), for whom “existence is being in so far as it is ‘grounded’, founded in a unique, universal Ground acting as its ‘sufficient Reason’” (Žižek, 2002: 60).

What we do next is transform the list of categories of ground into a system of logically inter-related categories.

ORGANISING THE CATEGORIES OF GROUND

In his rearticulation of the inter-relations between the classical logical categories[25] of the contrary and the contradictory, Greimas (1968) produces what has come to be called the *semiotic rectangle*, or *square*, which is a device that enables us to both develop as well as organise a set of related categories as a logical system. Starting with a single category, *S*, or even with an initial relation of contrariness, *S* and –*S*, Greimas shows how it is possible to generate an inter-related system of four terms to capture an extended semantic range of descriptions of the phenomenon of interest (for Greimas it was narrative that was the primary object interest). The symbols $\overline{S}$ and $-\overline{S}$ represent the contradictories of *S* and –*S*, respectively, and are organised as shown in Figure 1.

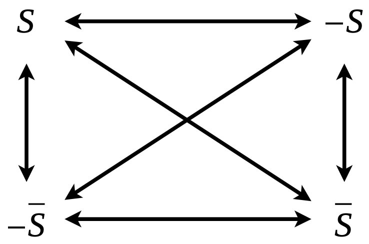

Figure 1: The Greimassian semiotic rectangle.

To animate any specific application of the semiotic square, categories have to be generated or related as binarily opposed. In our case we already have a set of four categories, so the game becomes one of organising those into a system of binarily opposed categories. This is, incidentally, a good test of the internal consistency of the implicit generative logic of a collection of inter-related categories.

Davis & Johnson (2007) noted in their observations of mathematical activity in classrooms that the predominant mechanism for the elaboration of mathematics in the schools participating in their research project was the use of worked examples. The utility of the worked example resided in its being a demonstration of the exemplary usage of some or other algorithm that students needed to learn to apply. The pedagogic belief supporting the use of worked examples appeared to rest on the assumption that, with sufficient exposure and practice, students will synthesise general principles from the examples and would thus be flexibly enabled to solve any problems typified by those worked examples. In different terms, the observed use of worked examples to teach mathematics appeared to be about training students to learn particular selections and sequencings of compositions of operations over collections of objects to solve particular classes of problems. Nothing new there, except to point out that the particular type of use of worked examples we refer to has the effect of setting mathematical activity as grounded in the algorithmic.

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[25] See, for example, Kneebone (1963: 3-25) for a discussion of the relations in classical logic. Greimas' semiotic rectangle is a familiar resource that is used extensively in semiotics, and the mathematics education reader should have no difficulty finding a suitable text discussing its use.

However, Davis & Johnson (2007) also drew attention to what appeared to be the use of exemplary solutions to problems construed as images as resources for regulating mathematical activity. The phenomenon was registered most explicitly in the mathematical activity of students, who often tended to use their teachers' exemplary solutions to problems as images for regulating their own work—at times even to the point of producing obviously nonsensical attempts at solutions to problems because they were so intent on following a teacher's solution in a manner dominated by an imagistic appreciation of the teacher's worked example as an exemplary solution (Jaffer, 2010; 2011). Once the endemic use of imagistic data is noticed, it then becomes fairly easy to start spotting the criteria used by teachers and their students and concerned with transforming the expressions that represent mathematical thought as though they were images. Suddenly a universe of criteria devoted to teaching students about the spatial transformation of expressions reveals itself as central to much of what unfolds as mathematical activity in schooling. This tells us that it's not unusual for mathematical activity in schools to be grounded in the iconic.[26]

So, on the one hand we have the use of algorithmic resources that select and sequence the series of operations apparently required for the solution of specific classes of problems. Described in this way, the algorithmic nature of mathematical processing in schooling is emphasised. But, on the other hand, often included amongst the problem-solving resources are resources for manipulating imagistic data, derived in the main from mathematical expressions.[27]

The use of imagistic data is intuitive because what something looks like is, quite literally, sensible, and so capable of fairly easy integration into experience. It's great shortcoming, however, resides precisely within its sensibility. Considering the iconic in general terms for the moment, we note, following Fodor, that icons don't have canonical decompositions since they're subject to what Fodor (2010: 173) calls the "picture principle": "If P is a picture of X, then parts of P are pictures of parts of X". This means that icons can be decomposed in whichever way we care to do so—which is very unlike the state of affairs that arises from the restrictions placed on discursive representations, like those produced by the lexico-grammar of mathematics. The latter representations are composed of expressions that ***do*** have canonical decompositions, and so can't be broken up in just any which way. Iconic ***ground***, which is different from the iconic in the general sense, is indexed by the—usually distorting—over-determining effect that arises when mathematical expressions are treated as a source of imagistic data for the purposes of mathematical processing. The

[26] See Johnson & Davis (2010) for a more extended discussion of examples of such.

[27] This state of affairs is very interesting because the algorithmic, as many mathematics educators often lament, can apparently be acquired and used without so-called "sense-making" by the student. We take such sentiments to be muddled because they register a failure to recognise that since syntactical processing can be accomplished without the student being able to integrate such processing into their intuitive appreciations of experience, that does not mean that their mathematical activity is "meaningless". In fact, we would argue that the syntactical is to be grasped intelligibly rather than sensibly—treating the distinction between the sensible and the intelligible in its full, Platonic, sense.

algorithmic and iconic grounding of mathematics that recur in pedagogic situations are, thus, always in a state of tension because they work in different, often antithetical, ways. Yet, where the sensible is privileged in pedagogic situations, the algorithmic is brought under the aspect of the iconic by means of the fashioning and prescription of combinatorial resources that transform mathematical expressions.

The discussion up to this point encourages us to take the distinction between algorithmic ground and iconic ground as our primary binary opposition in generating our system of categories. Algorithmic and iconic grounds are thus to be placed in a relation of contrariness in the Greimassian schema. To complete the system we need to specify the contradictories of algorithmic and iconic ground, and the two categories we have at our disposal for doing so are empirical and propositional grounds. To place the categories in the appropriate places in the system I proceed by emphasising something which is very pertinent to the relation of algorithms to propositions in mathematics, which will probably irritate those who hold on to and celebrate the antagonistic opposition between conceptual and procedural knowledge/thinking: the algorithmic is the apogee of propositional thought. Paradoxically, the algorithmic emerges at the point at which propositional thought has been distilled into a selection and sequence of formal rules that establish, with blind and absolute necessity, a desired mathematical outcome, evacuating all subjective engagement and also rendering all carping interlocutors silent. This tells us that propositional ground should be positioned as the support of algorithmic ground and in contradiction to iconic ground, which leaves empirical ground to take up the final position in opposition to algorithmic ground and in support of iconic ground. Are these arrangements reasonable? If we consider that the propositional is that which renders mathematics intelligible, and that the iconic is, fundamentally, sensible, then organising the categories so that the propositional and the iconic are contradictories is appropriate. Further, given that the algorithmic—in the way in which we have defined it here—is concerned with the formal production of mathematical necessity, and that the empirical is defined as that which is concerned with the generation of and arriving at an acceptance of results by way of inductive generalisation from measuring, it is appropriate to have the algorithmic and the empirical in a relation of contradiction. It is also appropriate to have the empirical as that which is supportive of the iconic, because the latter is but only a brute instance of the empirical.

Finally, the relation between the propositional and the empirical is one of tension because the empirical can display entities that register the muteness—and so failure—of the propositional at certain points because the empirical is insistent where the propositional lacks. The categories of ground can, therefore, be arranged to Greimas' model as displayed in Figure 2.

There are six pairs of binary relations discernable from the organisation of categories displayed in Figure 1: algorithmic-iconic, iconic-empirical, propositional-empirical, algorithmic-propositional, propositional-iconic, algorithmic-empirical. In the next section we develop additional resources for describing those six relations.

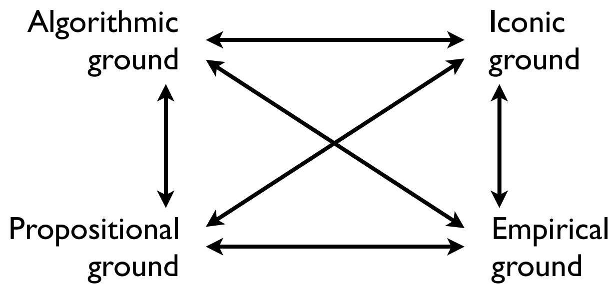

Figure 2: A Greimassian organisation of the categories of ground.

ORIENTATIONS TO PRIVILEGED TEXTS IN PEDAGOGIC SITUATIONS

In his discussion of the semiotics of culture, Lotman (1990) introduces twin concepts that are potentially productive for describing and thinking about the constitution of mathematics in pedagogic contexts. The concepts are: ***text-oriented cultures*** and ***grammar-oriented cultures***. What these orientations are for Lotman is captured with great clarity by Eco in his Introduction to Lotman's ***Universe of the Mind***:

> Cultures can be governed by a *system* of rules or by a *repertoire* of texts imposing models of behaviour. In the former category, texts are generated by combinations of discrete units and are judged correct or incorrect according to their conformity to the combinational rules. In the latter category, society directly generates texts, which constitute macro units from which rules can eventually be inferred, but which initially and most importantly propose models to be followed and imitated.
>
> A grammar-oriented culture depends on 'Handbooks', while a text-oriented culture depends on 'The Book'. A handbook is a code which permits further messages and texts, whereas a book is a text, generated by an as-yet-unknown rule which, once analyzed and reduced to a handbook-like form, can suggest new ways of producing further texts. (Eco in Lotman (1990: xi); italics in the original.)

All of us have at one time or another observed, even being subjected to, teaching that proceeds by insisting that the student repeat, precisely, the text produced as exemplary by the teacher. We may even, at times, have adopted just such a strategy for learning what we were being taught, transforming the teacher's discussion into a series of texts to be precisely rehearsed and repeated. Lotman's notion of text-orientation resonates with the sort of teaching and learning just described. His notion of grammar-orientation is suggestive of syntactical symbol manipulation and propositional descriptions of relations between mathematical objects. So there does appear to be a metaphorical resonance of Lotman's categories with some of the ways in which mathematical activity is thought about and discussed in mathematics education. However, while Lotman's distinction between different orientations appears to be potentially helpful, the suggestion that the use of combinatorial rules is somehow suspended, or reduced, in text-orientation is questionable. After all, as has been emphasised repeatedly by Chomsky since the 1960s, all language consists in the articulation of a finite system of rules for producing infinite combinations of finite collections of lexical elements (see Chomsky, 2009). On this issue Chomsky makes it clear that he was developing a line of thought already there in the work of Wilhelm von Humboldt in the 19th century (see Humboldt, 1971).

Shifting now from language to mathematics, we note that one characteristic of mathematics is that it is, just like language, combinatorial in the sense emphasised by Humboldt and Chomsky, and so it is expressed in the form of syntactic articulations on lexical elements (symbols and terms). This suggests that our unease with Lotman's too polarised definitions of grammar- and text-orientation with respect to language must be held to when considering the potential use of the distinction for descriptions of mathematical activity in pedagogic situations. If we follow the implications that flow from Humboldt and Chomsky, then the distinction between text- and grammar-orientation can't reside in the respective absence or presence of the use of combinatorial rules because the latter is a condition of possibility of language, of mathematics, and even of thought (see, also, Fodor 2010). In other words, *both* grammar- and text-orientation must entail the deployment of combinatorial rules, but must do so in different ways if they are to be thought of as distinct categories. So, how can Lotman's distinction between grammar- and text-orientation, which does appear to enjoy an intuitive truthfulness, be refashioned to be of use in mathematics education research?

Jaffer (2010) attempts to reconfigure Lotman's grammar/text-orientation distinction as a *context-independent/context-dependent criteria* distinction, focusing on the way in which evaluation functions in the production of texts in pedagogic situations. We can rework Jaffer's idea by noting that the evaluative criteria for the production of texts can take the form of two general types. In one type—what Jaffer refers to as *context-dependent*—there exist criteria specific to the particularities of the expressive features of the texts that are to be produced, treating those features as key to that which is to be *reproduced*, so that the combinatorial rules include the means for manipulating the expressive resources that are provided. For example, rules for shifting mathematical symbols around spatially are often explicitly invoked in such situations. Once we've formulated the matter in this way we can see that it is possible to suspend the condition that the criteria be necessarily context dependent, because combinatorial rules for manipulating expressions can be transported across contexts—which is not to say that in many empirical instances there are not criteria in use that *are* context-dependent. In the other form—*context-independent* to Jaffer—the criteria are concerned with effecting the use of general computational resources, in relation to which the expressive conventions for displaying mathematical work are positioned as secondary—perhaps even treated as epiphenomenal.

What we now have is a shift from Lotman's grammar/text-orientation distinction and Jaffer's context-independent/context-dependent criteria distinction to a distinction that fixes on the positioning of the expressive resources as, either, featuring amongst that which is of primary concern mathematically, or that which is secondary. Is this a reasonable transformation of Lotman's grammar/text-orientation distinction? Well, Eco (1979: 138) makes an interesting remark in which he indicates a second distinction drawn by Lotman, viz., a ***content-oriented/expression-oriented*** distinction:

> Lotman suggests that text-oriented societies are at the same time expression-oriented ones, while grammar-oriented societies are content-oriented. The reason for such a definition becomes clear when one considers the fact that a culture which has evolved a highly differentiated content-system has also provided expression-units corresponding to its content-units, and may therefore establish a so-called 'grammatical' system — this simply being a highly articulated code. On the contrary a culture which has not yet differentiated its content-units expresses (through macroscopic expressive grouping: the texts) a sort of content-nebula. (Italics in the original.)

Given the difficulties that inhere in the grammar/text-orientation distinction for us—because language and thought are taken to be combinatorial, hence "grammar-oriented"—the distinction between *content-* and *expression-orientation* is one that can be used to name the distinction in orientations to privileged texts in the pedagogic situations of the teaching and learning of mathematics.

CLOSED AND OPEN PEDAGOGIC TEXTS

Using the work of Eco (1984) and Bernstein (2000), Jaffer (2011) generated the categories *open* and *closed pedagogic texts*. Translating Jaffer's propositions into more mathematically precise terms, we can say that an open pedagogic text is one in which the properties of magmas and magma-like objects, as well as descriptions of the collections of objects that inhabit the magmas and magma-like objects, are made known to the student. Recall that a magma is an object constructed by taking an operation together with the set of objects over which it ranges, usually represented by the notation $(A,*)$, where * indicates the operation and A the set of objects that serve as arguments for *. A magma is subject only to closure.[28] It is knowledge of the properties of objects of the type $(A,*)$, of *, and of the objects that populate A, that enables the student's work to flexibly and freely converge on solutions to problems, or to formulate conjectures, or prove the mathematical truth of propositions. In other words, the use of the particular combinatorial resources of relevance to a situation is not rigidly selected and sequenced, but is nevertheless convergent on a particular result. With closed pedagogic texts there is little explicit concern with the properties of objects of the form $(A,*)$, or with the nature of either * or of the objects that serve as arguments for *. One common type of closed pedagogic text is represented by attempts to present the student with particular fixed selections and sequences of operations designed to realise very specific classes of outcomes. The particular selection and sequence of operations that participate in closed pedagogic texts usually require the expressions describing the particular mathematical situation of concern to

[28] The idea of taking a set together with an operation over the set as an object of study should be familiar to anyone who's been exposed to abstract algebra (there are numerous elementary texts easily available to those who haven't).

be in very specific forms. That is why teachers who exploit closed pedagogic texts invariably require their students to initially arrange expressions in some or other "standard form", so that the sequence of selected operations can begin to do its work. All attention is focused on the particular, prescribed selection and sequence of operations for achieving some or other end, and what the student is required to do is learn the particular selections and sequences of operations that enable the solution of various classes of problems. Where there is a need to deal with variation in problems of a certain class, teachers tend to treat the variations as distinct problem types and provide different selections and sequences of operations for each type. Jaffer (2011: 240-244) provides an extended analytic discussion of an example of a closed pedagogic text.

GROUND, ORIENTATIONS TO TEXT AND OPEN/CLOSED TEXTS

We can now bring the work of the previous sections together. First we need to consider orientation to privileged texts, taken together with texts as open or closed, are to be thought of when brought into relation with the categories of ground. Recall that the inter-relations of the categories, as depicted in Figure 2, announces six binarily construed relations: algorithmic-iconic, iconic-empirical, propositional-empirical, algorithmic-propositional, propositional-iconic, algorithmic-empirical. Each of the relations realises a blend of two categories of ground, and if we desire to describe those relations in terms of orientations to text and texts as open or closed, the descriptors we seek should be amenable to describing blends. With that in mind, we describe the orientations to text as realisable as strong or weak, and similarly for open texts and closed texts. We then have two states for each mode of textual orientation and for each of open/closed texts.

Let $O = \{O_c^+, O_c^-, O_e^-, O_e^+\}$ be the set of modes of textual orientation, content-oriented (c), (strong (+) and weak (-)) and expression-oriented (e), (strong (+) and weak (-)). Let $T = \{T_o^+, T_o^-, T_c^-, T_c^+\}$ be the set of modes of pedagogic texts, open (o), (strong (+) and weak (-)), or closed (c), (strong (+) and weak (-)). Taking the cross product of O and T, $O \times T$, produces the set of sixteen elements listed in Table 2, showing the possible combinations of textual orientations and open and closed pedagogic texts as determined by our selections of elements for O and T. Note that all that $O \times T$ shows is a list of potentially useful names, some of which may well be unintelligible when brought up against the empirical.

Pedagogic text: open/closed

Orientation to pedagogic text: content/expression $\times$	T_o^+	T_o^-	T_c^-	T_c^+
O_c^+	O_c^+/T_o^+	O_c^+/T_o^-	O_c^+/T_c^-	O_c^+/T_c^+
O_c^-	O_c^-/T_o^+	O_c^-/T_o^-	O_c^-/T_c^-	O_c^-/T_c^+
O_e^-	O_e^-/T_o^+	O_e^-/T_o^-	O_e^-/T_c^-	O_e^-/T_c^+
O_e^+	O_e^+/T_o^+	O_e^+/T_o^-	O_e^+/T_c^-	O_e^+/T_c^+

Table 2: The elements of the cross product $O \times T$

The question that now confronts us is: which of the sixteen elements of $O \times T$ should be associated with the pairs of categories of ground: algorithmic-iconic, iconic-empirical, propositional-empirical, algorithmic-propositional, propositional-iconic, algorithmic-empirical?

We consider each pair in relation to $O \times T$. Let the letters *A*, *I*, *E*, and *P* stand for algorithmic ground, iconic ground, empirical ground and propositional ground, respectively. For now, pairs made up from *A*, *I*, *E*, and *P* are taken as non-ordered, so that, e.g., (*A*,*E*) is the same thing as (*E*,*A*), and so forth for all the other pairs.

(1) *Algorithmic-Iconic*: this relation is exemplified by those instances in which the iconic plays a strong regulative role in the selection and organisation of combinatorial resources. The text is strongly closed because no attention is given to the properties of objects of the form (*A*,*) and the textual orientation is characterised by a strong focus on the imagistic features of expressions. Therefore, (*A*,*I*) ⇒ O_e^+/T_c^+. (2) *Iconic-Empirical*: here the strong expression orientation of the iconic is tempered by the intrusion of the empirical, also generating thereby a weakening of the closed nature of the iconic. Therefore, (*I*,*E*) ⇒ O_e^-/T_c^-. (3) *Propositional-Empirical*: the propositional always drives activity in the direction of content orientation and textual openness, but the presence of the empirical weakens both the openness of the text and the orientation to content because it inserts the potential for induction, diluting the force of the propositional. Therefore, (*P*,*E*) ⇒ O_c^-/T_o^-. (4) *Algorithmic-Propositional*: the combination of the propositional and the algorithmic presents the strongest realisations of both textual openness and content orientation. Here the combinatorial resources are intelligibly realised. Therefore, (*A*,*P*) ⇒ O_c^+/T_o^+. (5) *Propositional-Iconic*: we can think of this oppositional relation as an instance in which attempts are made to render the propositional intelligible by means of the iconic, like in so-called “proofs-without-words”, or, as is suggested by Whitehead’s (1911: 61) charming description of the exploitation of the imagistic features of notation as “reasoning almost mechanically by the eye”. Content orientation is displaced by a strong orientation to expression, but the text nevertheless retains some openness because the of the intelligibility provided by the propositional. Therefore, (*P*,*I*) ⇒ O_e^+/T_o^-. (6) *Algorithmic-Empirical*: the intrusion of the empirical into the algorithmic suggests those moments at which the algorithmic is either as yet not able

to do its work, or at which the algorithmic has arrived at a point of impossibility. In either case content orientation is weakened because there is a content absence, and the closure of the text is heightened since the empirical is a stand in for the absent content. Therefore, $(A,E) \Rightarrow O_c^- / T_c^+$.

CONCLUDING REMARKS

The descriptions of the six binarily construed relations just derived are indicated graphically in the extended semiotic rectangle shown in Figure 3. I shall, at this time, leave open the question of whether it's reasonable to propose that the implications ($\Rightarrow$) of the six formulae be substituted by equivalences ($\Leftrightarrow$). I'll take up that story in another essay. Due to limitations on space we cannot provide an adequate demonstration of the use of the analytic device constructed here but we can do so in a conference presentation. There is, however, one final issue to be dealt with which may be of help in that regard, and that is the question of how the device connects with empirical instances of the constitution of mathematics in pedagogic situations. A description of the operational activity that unfolds in pedagogic situations names three things: (1) the operations that emerge in the situation, as well as (2) the collections of objects over which the operations range, and also (3) the criteria governing the selection and sequencing of the operations in play (Davis, 2011). The criteria that are named in turn indicate the ways in which the mathematical activity in a pedagogic situation is regulated. Recall that the categories of ground are descriptions of the regulation of mathematical activity, so the link between the operational activity of agents and the device represented in Figure 3 is established by marking out the regulative criteria.

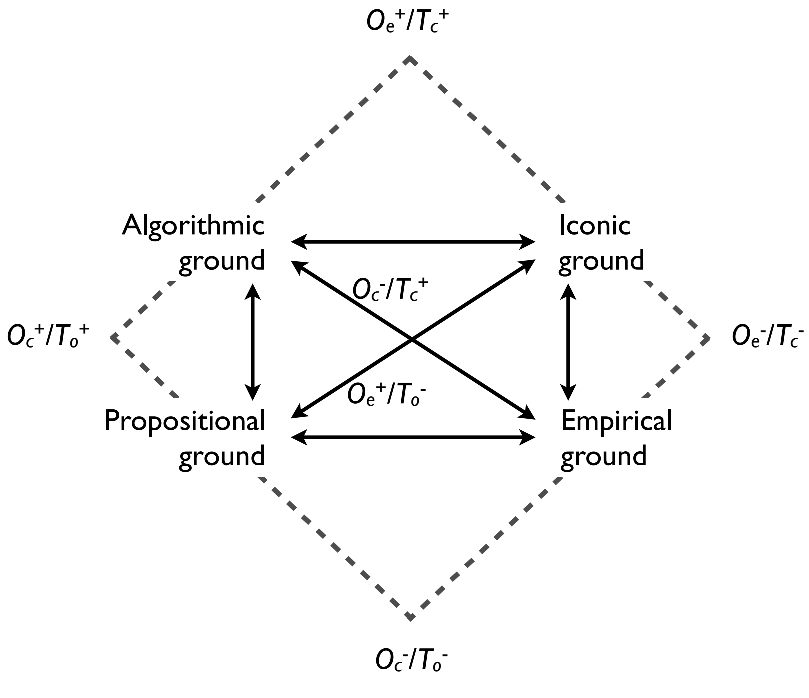

Figure 3: The categories of ground related to orientations and open/closed texts.

The types of ground, considered as pairs, connect with the relevant binarily construed relations displayed in Figure 3, indicating the orientations to text of agents in the situation as well as whether or not pedagogic texts are realised as open or closed.

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REMARKS ON RECENT USES OF THE TERMS *OPERATIONS*, *OBJECTS* AND *DOMAINS* IN LOCAL DESCRIPTIONS OF MATHEMATICS TEACHING

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In this paper I examine the use of the terms object, operation *and* domain *in recent work reported on in Venkat & Adler (2010) and in Adler & Venkat (2011). Venkat and Adler take up elements of a methodology for describing the constitution of mathematics in pedagogic situations, developed by a research group based at the University of Cape Town, for the purposes of discussing and analysing the ways in which procedures are used in the teaching of school mathematics. In developing their argument, Venkat and Adler use the terms* object, operation *and* domain *in ways that are not fully aligned with the mathematical meanings of those terms, thereby disrupting the very methodology they seek to borrow from. Using the terms, they derive the category of* domain violation *and its sub-categories (*domain reduction, domain constraining, domain disconnection *and* domain misrecognition) *for describing features of "procedural approaches to mathematics teaching". We show that Venkat and Adler's determinations of the proposed category of* domain violation *and of its sub-category of* domain reduction *are internally inconsistent by re-examining the archive of information from which they produce the data informing their construction of the categories.*

INTRODUCTION

In their presentations to the 2010 conference of the Kenton Education Association and to the 2011 SAARMSTE conference, Venkat & Adler (2010) and Adler & Venkat (2011), respectively, announced an intriguing set of four categories for describing what they referred to as the phenomenon of *domain violation*, encountered in "procedural approaches to mathematics teaching in the South African context". What I do here is interrogate Venkat & Adler's (2010) and Adler & Venkat's (2011) proposed notion of *domain violation* and one of its sub-categories, *domain reduction*. I do so for various reasons, three of which are highlighted here.

First, because the argument developed by Venkat & Adler (2010)/Adler & Venkat (2011) has, at first blush, an apparent affinity with the elements of another body of work[29] being developed, but is, in reality, very different from that work. The circulation in the literature of bodies of work, arguments and ideas that are apparently closely related yet different, and that employ many of the same terms, is a situation

[29] A sample of which is: Arendse (2011), Basbozkurt (2010), Chitsike (2011), Davis (2010a, 2010b, 2011), Gripper (2011), Jaffer (2009), Mackay (2010).

ripe for confusion. It is confusing not only for experienced researchers in the field, but even more so for graduate students of mathematics education. There is, therefore, a general need to draw out the central differences between the two positions.

Second, I wish to emphasise the point that no matter what our particular research foci are when researching the teaching and learning of mathematics, we have to keep in mind the fact that we are talking about some or other phenomena ***always in relation to mathematics***. Venkat & Adler (2010)/Adler & Venkat (2011) use terms like *operation* and *domain* in ways that diverge somewhat from the usual mathematical meanings of those terms, which then raises the question of whether or not their descriptions of mathematical activity generally are mathematically attuned.

Third, given the general mathematical features of operations—which I will detail later—we do need to ask the following question of Adler and Venkat: in which senses do shifts from one domain of operation to one or more different domains constitute one or other kind of ***violation***?

From this point on I shall refer to Venkat & Adler (2010)/Adler & Venkat (2011) as VA/AV.

VENKAT & ADLER ON OPERATIONS, OBJECTS & DOMAINS

In Davis (2010c, 2010d, 2011) I draw out a series of propositions used to orient the production of descriptions and analyses of the constitution of mathematics in pedagogic situations: (1) the operations that populate mathematics are functions, which means that operations can be thought of as a subset of the set of n-tuples constituted by taking the cross product of the domains that supply the arguments and the codomain of the operation; (2) since there can, in principle, be an infinite number of different rules associated with a given function—which is just a particular subset of n-tuples—we can say that the processes associated with any given operation can, in principle, be infinitely various; (3) the correlation of signifier and signified, as understood in semiotics, is arbitrary—that is, not necessary—so opening the reading of any given signifier to the possibility of semantic variation, which means that readings of the signifiers of mathematics in pedagogic situations do not entail necessary associations of specific contents, and so contents other than those we usually expect in response to a given signifier can emerge in pedagogic situations; (4) given the possibility of semantic variation, pedagogy is necessarily evaluative in that it always attempts to fix the correlation of specific expression-content realisations and polices attempts at such realisations.

What the four propositions tell us, amongst other things, is that it is perfectly possible for mathematical computations to be realised in very many different ways, and not only with respect to the number of "steps" employed but also, more importantly, in terms of the very processes and the stuff being processed.

Operations and objects *à la* Venkat and Adler

In their presentations, VA/AV describe an *operation* as: "any action or step that acts on an input object. Singly, or in sequences of ordered operations, these steps constitute procedures which produce transformed objects". Now, what constitutes a "step" is, ultimately, context dependent since it has no essential nature. An "action", however, suggests a particular, recognisable moment in a process and so is suggestive of something akin to an operation. VA/AV do not include the notion of function in their predication of the term "operation", and consequently, what they refer to as an "operation" in any particular instance may, or may not, be an operation in the mathematical sense of the term. Further, the term "operation", in its construal as a "step" by VA/AV, is referred to as both a single entity as well as a series of operations.

What a "transformed object" might be is not specified in any way by VA/AV. Are they implicitly proposing that descriptions of mathematical objects are similar to folk descriptions of natural kinds (Kripke, 1980; Arendse, 2011) rather than definite descriptions? And, in what sense can a mathematical object be transformed for VA/AV? Let's address these questions first for objects considered individually. On the one hand, if we change any of the essential features of a mathematical object in our descriptions of the object, we do not transform the object but rather constitute an entirely different mathematical object, or even no mathematical object at all. For example, if we change any of its essential properties in our description of an object that conforms to the notion *square*, say the sizes of the angles at vertices, we no longer have an object that we can refer to as a *square*. On the other hand, if we alter features that are non-essential in the general sense, like the lengths of the line segments that, partly, constitute a specific square, we get another object, which nevertheless remains a member of the category *square*.

Let's now consider the idea of a "transformed object" in the context of sequences of operations, which could also be described as series of nested operations (in the mathematical sense of *operation*). From VA/AV's descriptions of objects and operations it seems that a "transformed object" is something produced by a procedure, by which they appear to be referring to the intermediate and ultimate codomain objects selected by a nested series of operations constituting a procedure. With respect to transformations in the context of procedures, it has repeatedly been pointed out that changes at the level of expression are legitimate, even required, while simultaneous stasis, or identity, is required at the level of value, both locally and globally (Davis, 2010a, 2010b, 2011). In other words, transformations in the context of procedures entail both the production of difference (at the level of expression) and the simultaneous reproduction of identity (at the level of value), as sequences of equivalent expressions or equations are produced in generating a solution or argument. VA/AV's use of the term "transformed object" appears to be a reference to the level of expression, but perhaps they're taking it as understood that identity is preserved at the level of value. Then again, they don't appear to be explicitly concerned with the distinction between levels of expression and value. Perhaps VA/AV's use of the term is intended to capture a general intuition along the lines of: *that which exists now looks different from that which occupied the same place before*? Whatever the case may be as regards their awareness of the distinctions described here, VA/AV remain imprecise on what a "transformed object" is.

VA/AV's imprecise deployment of the terms "object", "operation" and compound terms, like "transformed object", suggests to us that they think about mathematical objects in a manner more suited to the rigid designations used for referring to natural kinds, where objects can be altered substantially without effecting changes to their names/category membership (cf. Kripke, 1980).

Domains *à la* Venkat and Adler

There are a number of different ways in which the term *domain* is used in mathematics. First, *domain* is used to reference a collection of objects that can legitimately serve as arguments for operations and for the collections of objects over which more general functions and relations range. Second, *domain* is also used in the sense of a *universe* of a model or structure, referring to the class of objects that are of interest when considering the model or structure. Third, there are other uses of the term that are not of immediate concern to the point we are discussing here, like its use in the compound term, *integral domain*, viz. a ring, commutative under multiplication, having an identity element and no divisors of zero. VA/AV, however, claim to think of a *domain* as a *topic*: "We nevertheless observed key differences across the episodes, linked to the nature of the relationship between objects and operations within the 'domain' (or topic) within which the teaching episode was located;" and: "Through our elaboration of procedural practice we argue that there is a need to focus more closely on mathematical domains (topics) and the objects and operations within them in our work with mathematics teachers." They attempt to define two species of domain—*enacted domain* and *intended domain*: "'enacted domain' refers to the class of objects referenced by unfolding operational activity in our episodes," aver VA/AV, while "'intended domain' is indexed by an input object that would regulate operational activity in ways that have validity within the mathematical community."

Let's deal with their use of the term "enacted domain" first, which appears to be identical to the first mathematical use of the term spelt out earlier. At least, that is the case if the statement "objects referenced by unfolding operational activity" is actually referring to the arguments of the operations that populate a procedure. But then, again, the full range of "objects referenced" are generally more extensive than the collections of elements supplying the arguments to operations because there are objects that function in a regulative manner on mathematical activity, but which are not arguments of the operations in play.

The stipulation of the "intended domain", which is "indexed by an input object that would regulate operational activity in ways that have validity within the mathematical community", fares no better. With their use of the term "input object" they appear to be referring to objects that function as arguments to operations, but they also appear to want such an object to have an overall regulative effect on mathematical activity. So we land up in just about the same place we did when considering "enacted domains".

The major difference between the two proposed domain types seems to be held in the requirement that the objects of the "intended domain" regulate activity "in ways that have validity within the mathematical community". Now, when it comes to specialised knowledge, requiring that validity be what is decided by a "community" is problematic, both within and outside of mathematics. Which "mathematical community" is of relevance here? Only mathematicians? Or are mathematics educators also included? What about teachers of mathematics? Or their students? And what about users of mathematics in industry and commerce, and elsewhere? All of these groups could propose competing claims about what "mathematical validity" is. We know, for example, that what is often claimed as valid mathematics in schooling can have weak, or even null, validity in many other contexts. Think of the ways in which many mathematical notions are defined in schooling so that they support some or other pedagogical end, but which are mathematically invalid. An obvious candidate for an idea that is spectacularly wrecked in schooling in such a manner is the idea of a function; another is the idea of an equation. Nevertheless, in very many schools, and perhaps even in the school curriculum statements and in teacher training, both the inappropriate uses of mathematics with respect to extra-mathematical contexts and wrong conceptions of mathematical notions (like functions and equations) *are* often accepted as valid.

The problems that beset VA/AV's use of the terms *operation* and *domain* appear to derive directly from them not using those terms in a manner consistent with mathematics. Nonetheless, let's see if VA/AV's metaphorical usage of the terms can support the secondary category of *domain violation* and its proposed sub-categories.

DOMAIN VIOLATION

To get a grip on *domain violation* we need to have a look at supposed instances of such as discussed by VA/AV because they use empirical instances to exemplify the sub-categories they develop. The technical terms of the four sub-categories of domain violation proposed by VA/AV are: *domain reduction*, *domain constraining*, *domain disconnection* and *domain misrecognition*. The stipulations for each category are suggested in the descriptions of the types of domain violation exemplified by each of four different empirical instances of pedagogic and/or student activity. Here we shall discuss only the category of *domain reduction* because of limitations on space, but that is sufficient to draw out additional problematic features of VA/AV's proposed categories.

Nash and a putative case of *domain reduction*

In their explication of their category of *domain reduction*, VA/AV use an extract from a transcript of the third lesson of a series of seven lessons, reported on in Pillay (2006a, 2006b):

Nash:	… first make your x equal to zero … that gives me my y-intercept. Then the y equal to zero gives me my x-intercept. Put down the two points ... we only need two points to draw the graph.
Learner 1:	You don't need all the other parts?
Nash:	[…] What's important features of this graph? … we can work out … from here [points to the graph drawn] we can see what the gradient is … is this graph a positive or a negative?
Learners:	[Chorus.] positive.
Nash:	It's a positive gradient … we can see there's our y-intercept, there's our x-intercept [Points at the points (0;-3) and (2;0), respectively.].

[After a brief discussion on the labelling of points on a graph, Learners 2 and 3 engage Nash.]

Learner 2:	Sir, is this the simplest method sir?
Learner 3:	How do you identify which side must it go, whether it's the right hand side. [Nash interrupts.]
Nash:	[Responding to Learner 2.] You just join the two dots.
Learner 2:	That's it?
Nash:	Yeah … the dots will automatically … if it was a positive gradient it will automatically … if this was [Referring to the line just drawn.] negative … that means this dot [Points at the x-intercept.] will be on that side [Points at the negative x-axis.] … because if the gradient was negative, how could it cut on that side? [Points at the positive x-axis.]
Learner 2:	Is this the simplest method sir?
Nash:	The simplest method and the most accurate ...
Learner 4:	Compared to which one?
Nash:	Compared to that one [Points at the calculation of the previous question, where the gradient and y-intercept method was used.] because here if you make an error trying to write it in y form … that means it now affects your graph … Whereas here [Points at his calculations on the dual intercept method.] you can go and check again … you can substitute … if I substitute for 2 in there [Points at the x in 3x – 2y = 6.] I should end up with 0.

According to VA/AV, Nash's privileging of the dual-intercept method for the graphing of *linear functions*, in response to questions from his students, produces a *domain reduction* because the dual-intercept method, so they argue, cannot be used to derive the data that would enable the graphing of vertical and horizontal lines.

> The method was presented as general – was and promoted as efficient, error free and always applicable. Yet this method excludes lines of the form y = k or x = k. We thus categorized it as procedural, but also reductive of the construct of a line, and so the domain of all possible straight

> lines. We named this type of procedural teaching: **domain reduction**. (Adler & Venkat, 2011: 3. Bold in the original.)

So we see that for VA/AV this means that the "domain" is reduced to one that excludes vertical and horizontal lines.

The dual intercept method does not imply a *domain reduction*

Consider the general equation for a line: $Ax + By + C = 0$, where *A*, *B* and *C* are constants, and *x* and *y* are variables, with $A, B, C, x, y \in \mathbb{R}$ (not both *A* = 0 and *B* = 0). In the set of all possible lines in the Cartesian plane, there are lines corresponding to the case where *A* = 0 but *B* ≠ 0, and others corresponding to the case where *A* ≠ 0 but *B* = 0, and still others corresponding to cases where *C* =0. In the first two cases the general equation becomes, respectively, (1) 0·*x* + *By* + *C* = 0 and (2) *Ax* + 0·*y* + *C* = 0. Setting = 0 in (1) produces the equation $y = -\frac{C}{B}, B \neq 0$. Putting *y* = 0 we get the equation *C* = 0, which is true only if *C* has a value of zero. It is clear that no matter what value *x* takes on, the value of *y* is $y = -\frac{C}{B}, B \neq 0$. So the line is $\left\{(x, y): x \in \mathbb{R}, y = -\frac{C}{B} \in \mathbb{R}, B \neq 0\right\}$, from which we can select any two points, one of which can be $\left(0, -\frac{C}{B}\right)$. Similarly for (2). When *B* = 0, the dual intercept method shows that for $\forall y \in \mathbb{R}, x = -\frac{C}{A} \in \mathbb{R}$. This means that the line we seek is $\left\{(x, y): x = -\frac{C}{A} \in \mathbb{R}, y \in \mathbb{R}, A \neq 0\right\}$, and we can select any two points from that set to graph the line, including $\left(-\frac{C}{A}, 0\right)$. In case (3,) where *C* = 0, we see that *Ax* + *By* + *C* = 0 reduces to *Ax* + *By* = 0, producing $\left\{(x, y): y = -\frac{A}{B}x, x \in \mathbb{R}, -\frac{A}{B} \in \mathbb{R}, B \neq 0\right\}$ as the line of interest. Here, taking either *x* = 0 or *y* = 0 produces (0,0) as an intercept, and an additional point must be selected from the set describing the line.

If we take the set of all lines in the plane as our domain of interest, which seems to be what VA/AV intend for us, then there is no mathematically necessary reduction of the domain implied by the use of the dual intercept method, even though we are left with only one intercept in each of the three cases. This demonstrates that *the dual intercept method does not entail a mathematically necessary exclusion of the subsets* $\left\{(x, y): x \in \mathbb{R}, y = -\frac{C}{B} \in \mathbb{R}, B \neq 0\right\}$ *and* $\left\{(x, y): x = -\frac{C}{A} \in \mathbb{R}, y \in \mathbb{R}, A \neq 0\right\}$ *of* $\mathbb{R} \times \mathbb{R}$ *from the set of all lines in the plane*, as was claimed to be the case by VA/AV. This means that if Nash's treatment of lines in terms of the dual intercept method does, in fact, produce a reduction in the subsets of $\mathbb{R} \times \mathbb{R}$ that are lines, then it's not because of something intrinsic to the idea of a procedure designed to search for two intercepts that produces the distortion.

Since Nash's general topic across the seven lessons was, indeed, *linear functions*, there *is* a reduction of the set of possible lines in the plane to a set that excludes all lines that take the form $\left\{(x,y): x = -\frac{C}{A} \in \mathbb{R}, y \in \mathbb{R}, A \neq 0\right\}$, of which there are $2^{\aleph_0}$ in number. But such reduction clearly has nothing to do with the putative effects of the dual intercept method, as argued by VA/AV, on the domain consisting of the set of lines in the plane.

Nash's central organising idea

It may have been more productive for VA/AV, in their quest to illuminate the nature of "procedural approaches", to consider Nash's approach to the teaching of lines in finer detail. Earlier in the series of lessons, at around ten minutes into the first of seven lessons, Nash clearly draws out a proposition that is central to his intended programme: *all one needs are two appropriate points to draw a specific line*. He also makes it clear that all the different procedures for graphing lines entail finding two appropriate points to use for that purpose. Interestingly, VA/AV don't refer to the extract, shown below, and it is clear from the extract that Nash thinks about his teaching programme on lines in relation to a central existential feature of lines.

Nash: So, the definition of any straight line will be what?

Learners: [Silent.]

Nash: The shortest distance between two points. … So how many points do I need to draw a straight .. the line?

Learner: Two.

Nash: Two. … As long as I got two points. … I can draw a straight line between any two points. … Write it down somewhere. [Nash writes his definition of a straight line on the board while the learners are busy copying work from the board.] … So, at the back of our mind we know although I'm going to use five points, there is a way to draw the graph if I just had to take two of these points [Indicates the plotted points.] I could have drawn the same straight line. … You will see later on as we continue we're going to learn different methods. This method is actually referred to as the table method. Then you'll see later on we gonna use different methods that will make it easier for us to draw. Where we're only looking for two points. If I can just get the two points, I can join the two points and I'll get my linear function.

Nash explicitly refers to the idea of using two points to draw a line *more than once in each of the seven lessons*, and *in excess of fifty times across the seven lessons*. That a couple of students failed to get the message in a way that enabled them to recognise the particular function of the different types of data produced by the different line graphing procedures is interesting, suggesting that we should examine Nash's use of pedagogic evaluation. One feature of his teaching that is immediately apparent upon reading the lesson transcripts is the paucity of instances of his interrogation of students' grasp of ideas. When he does ask questions Nash tends to answer them himself, perhaps in an attempt to model the ways in which his students should respond. The way in which Nash teaches largely forecloses the public emergence of students' thinking, and since his students' thinking is not available to public scrutiny and interrogation, it is little wonder that a few of them fail to acquire the means to formulate their ideas appropriately.

By locating and drawing out Nash's central organising idea we can see that the use of any of the procedures for drawing lines that he teaches is to be regulated by the idea that two points are needed to draw a line. In the event that only one point is produced, an additional point is to be sought.

Nash's treatment of the dual intercept method

Let's now turn to Nash's version of the dual intercept procedure in its detail, the pertinent bits of which are exhibited in the following extract from the third lesson: "First make your x equal to zero. That gives me my y-intercept. Then the y equal to zero gives me my x-intercept. Put down the two points. We only need two points to draw the graph." As can be seen from the extract, Nash summarises the method succinctly. VA/AV make no reference to this extract, which captures a part of the lesson that unfolds prior to that shown in the extract that they do use. Nash motivates for the study of the dual intercept method by describing the previous method studied—the gradient/y-intercept method—as requiring more labour, including precision in the drawing of the x- and y-axes.

Later in the same lesson Nash and his students engage with the following equations as they practice using his version of the dual intercept method: $x + y - 2 = 0$, $2x - 3y - 6 = 0$, $2x + 2y + 5 = 0$, $x - 2y + 2 = 0$ and $x - 2y - 2 = 0$. All the equations are of the form $Ax + By + C = 0$, but Nash does not discuss the general equation or the cases of lines where $A = 0$ while $B \neq 0$, or $A \neq 0$ while $B = 0$. Nash's version of the method therefore fails to indicate to his students that both the x- and y-intercepts are not always available. So it could well be that a number of his students implicitly introduce into Nash's treatment of the method an assumption that *all* lines are of the form $Ax + By + C = 0$ where $A \neq 0$ *and* $B \neq 0$. To the extent that many students insert such an assumption or its equivalent, they inductively construct an invalid generalisation.

Nash's use of his students' prior knowledge of lines

We should not forget that students' experiences of lines in a wide range of educational and other contexts precedes their encounter with the line as an equation, and that those experiences are what serve to partly ground the notion of a line while students become more familiar with the idea of a line as an equation. In fact, Nash attempts to exploit his students' intuitive experiential knowledge of lines by repeatedly drawing their attention to needing only two points to draw a line. Since he does not prove that to be the case in any of the seven lessons devoted to the topic of graphing lines, he is clearly relying on his students' intuitive experiences of lines. So, it would not be unusual to find that Nash's students accept that vertical and horizontal lines are, indeed, valid lines, but that many of them also believe that *all* lines can take on the form $Ax + By + C = 0$ where $A \neq 0$ *and* $B \neq 0$.

Do we have an instance of "domain reduction" here? Well, the question is unanswerable if we buy into the "communities of practice" idea—which is what appears to be suggested by VA/AV's appeal to the "mathematical community"—because what is registered as the valid treatment of lines is, arguably, volatile across the various fragments of what might be considered as the general "mathematical community", yet apparently quite stable internal to any particular fragment of the "community".

Nash and the national curriculum statement

Nash teaches his students a range of procedures for graphing lines—the "table method", the "gradient-intercept method" and the "dual intercept method"—and the different procedures are to be used in different problem contexts. The manner in which Nash teaches depends heavily on the use of worked examples and the global features of a relation like $Ax + By + C = 0$ (not both A and B zero) are rarely discussed explicitly. Therefore, to the extent that they register a need to have a more general grasp of the topic, his students have to synthesise the general ideas from their engagement with specific problems. From this we can see that lines of the form $y = k$ and $x = j$ would be encountered as specific graphing problems in the context of Nash's mathematics lessons. However, we must also note that the national curriculum, to which teachers like Nash are subject, curiously appears to exclude the explicit study of general relations and explicitly focuses on functions, so that lines are treated as linear functions, thereby excluding lines of the form $x = j$ from consideration when the topic of study is the line. Lines of the form $y = k$ are, presumably, covered by the form $y = ax + q$ which is indicated in the curriculum statements, but that's not at all obvious because teachers and textbook writers would have to consider the case when $a = 0$ in $y = ax + q$. That said, the curriculum does require that asymptotes, which are restricted to vertical and horizontal lines, for certain functions be indicated in sketches, and also that the derivative of the function $f(x) = b$ be studied in Grade 12, so that lines of the form $y = k$ and $x = j$ are required as resources at times.

So what is the "intended domain", according to the NCS? We conclude that the manner in which the national curriculum presents the content on lines already excludes lines of the form $x = j$ by prescribing the studying of lines as *linear functions*, and also that it, effectively, suggests the exclusion of lines of the form $y = k$ when lines are studied. The curriculum then also requires the use of such lines as asymptotes when graphing functions other than lines. This means that many teachers, in following the curriculum, will present the study of lines without considering the lines $y = k$ and $x = j$, but would have their students draw horizontal lines through (0,*k*) and vertical lines through (*j*,0) to indicate vertical and horizontal asymptotes as they are required, probably without any class discussion of the nature of those lines in relation to the topic of lines in general.

If we restrict ourselves to the topic of lines as specified by the NCS, then Nash's coverage of the topic is in general accord with the curriculum specifications.

CONCLUDING COMMENTS

So, does Nash's teaching of the dual intercept method present us with an instance of *domain reduction* in VA/AV's sense? With respect to the work it's meant to do, the category is internally inconsistent since the stereotypical empirical instance meant to exemplify it is easily shown not to entail a reduction; consequently, as it is currently formulated, the category has an empty extension.

The proposed category of *domain reduction* also indicates a lack of recognition by the researchers that objects and operations different from those usually taken as indexed by specific mathematics topics often circulate in the pedagogic situations of schooling, in two senses. There are instances when alternate objects and operations are used by teachers, but which are familiar mathematical entities (see Davis (2011) for an example). Then there are also instances of objects and operations or operation-like manipulations, that are not usually recognised as mathematical entities, being used (see Basbozkurt (2010); Davis (2010a) for examples). In both cases the objects in use, or implied, are not the same as those indexed by the topics being studied, and so we don't have reductions in domains but rather shifts to altogether different domains.

The general problem with the notion of *domain violation* and its sub-categories stems, ultimately, from VA/AV not using the notions of *operation* and *domain* in a manner consistent with mathematics. Consequently, they miss a few crucial, primary things about pedagogic situations—one being the specific features of the operations being deployed; another being the features of the objects used as arguments by operations; and also that the ordered ***n***-tuples that are operations can, in principle, be arrived at in an infinite number of different ways. The last point, especially, suggests the question: what it is that actually constitutes a ***violation***, and of what, exactly?

ACKNOWLEDGEMENTS

Thanks to Vasen Pillay for generously making his transcripts of Nash's lessons on lines available, and also to Hamsa Venkatakrishnan and Jill Adler for kindly making their Kenton 2010 conference presentation slides available. Thanks to Nicole Arendse, Megan Chitsike, Jaamia Galant, Derek Gripper, Ursula Hoadley, Piet Human, Shaheeda Jaffer, Cyril Julie, Caroline Long, Roger Mackay, Setseetso Matobako and Vasen Pillay for their comments on an initial draft of the paper. This paper arises out of the work of the *Group for the Study of Pedagogic Operatory*

Spaces on the Mathematics and Science Education Project at the University of Cape Town, funded by the Royal Netherlands Embassy. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not reflect the views of the Royal Netherlands Embassy.

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THE RELIABILITY OF A RESEARCH INSTRUMENT USED TO MEASURE MENTAL CONSTRUCTS IN THE LEARNING OF CHAIN RULE IN CALCULUS

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This paper reports on a pilot study conducted to validate a questionnaire designed to explore the conceptual understanding displayed by first-year University of Technology students in learning the chain rule in calculus using APOS (Actions-Processes-Objects-Schema) Theory. The reliability of the research instrument based upon the types of objects constructed by students when learning the chain rule is discussed. This was with the view to identify ways to assist with clarifying their understanding of the composition of functions and derivative. The analysis of written responses and interviews suggested that the instrument provided substantial information for identification of certain mental constructions that the research suggest worthwhile to consider.

INTRODUCTION

Gordon (2005) asserts that the chain rule is one of the hardest ideas to convey to students in calculus. This implies that while the chain rule is straightforward to write out, as simply stating that if $g(x)$ is a function differentiable at *c* and *f* is a function differentiable at $g(c)$, then the composite function $f \circ g$ given by $((f \circ g)(x)) = f(g(x))$ is differentiable at *c* and that $((f \circ g)^{'}(c)) = f'(g(c)).g'(c)$, or in Leibniz notation, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$, teaching and learning the chain rule is more complex. This rule tells us how to differentiate a function formed by the composition, or sequential application, of two simpler functions. The key to understanding the Chain Rule is to think about a function as a process which transforms one number into another. When dealing with a composition of functions, its best to first rewrite the definitions of the functions, using variable names which represent the different stages in the process. The derivative $\frac{dy}{dx}$ in simple terms measures are rate of change. Rate refers to ratio which implies division. The derivative indicates a special ratio of the form change in *y*/change in *x*. This can be described in Leibniz notation, which states that if $y = f(u), u = g(x)$ and *y* and *u* are differentiable functions, then, $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$.

The complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum. Dubinsky & McDonald (2001) suggested that APOS theory is a tool that can be used objectively to explain students' difficulties with a broad range of mathematical concepts and to suggest ways that students can learn these concepts. They further suggests that this theory can also point us towards pedagogical strategies that lead to marked improvement in (1) student learning of complex or abstract mathematical concepts, and (2) students' use of these concepts to prove theorems, provide examples, and solve problems. APOS proposes, in the form of a genetic decomposition, a set of mental constructions that a student might make to learn the concept of the chain rule and accessing it when needed.

The initial genetic decomposition proposed for this study indicates that: (1) A student with a function schema has developed a process or object conception of a function and composition of functions. (2) For a derivative schema, (i) He or she has developed a process conception of differentiation (ii) The student then uses the previously constructed schemas of functions, composition of functions and derivative to define the chain rule. In this process the student has recognizes a given function as the composition of two functions, take their derivatives separately and then multiply them. (iii)The student recognizes and applies the chain rule to specific situations.

The questionnaire was designed to: (1) predict mental constructions students might make in learning the chain rule, (2) find out how much of the chain rule they seem to be learning and using, (3) compare the mental constructions the students appear to be making with those indicated in the initial genetic decomposition, and (4) determine the limits of student knowledge.

Individuals were then selected for interview based on their responses to the written instrument. A full range of understanding was accessed by selecting participants who gave correct, partially correct and incorrect answers on the written instrument. The interviews were important to validate the students' written responses, i.e. the correlation between the status of the written responses and the level of understanding.

THEORETICAL FRAMEWORK

The research instrument was designed according to a specific framework for research and curriculum development in advanced mathematics education. This framework modelled in Figure 1 guided the systematic enquiry of how students acquired mathematical knowledge and what instructional interventions could contribute to student learning.

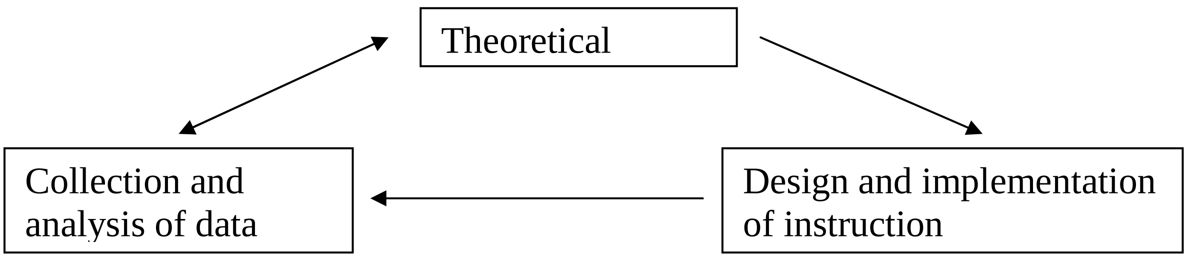

Figure 1: Paradigm: General Research Programme

The theoretical analysis served to propose mental constructions (the genetic decomposition) which the students might make in learning the chain rule concept (Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John,D., Tolias, T., &Vidakovi´c, D. , 1997). APOS theory is based on an overall perspective of what it means to learn and know something in mathematics. Asiala et al (2004, p.7) note that: 'An individual's mathematical knowledge is his/ her tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.'

These mental constructions and reconstructions can be described as follows:

- Action: Refers to a repeatable physical or mental manipulation as a reaction to external cues that give precise details on what steps to take. For example, students who interpret the idea of a function as contained in the 'formula' for computing values is restricted to the action concept of function (Dubinsky, 1991a).
- Process: When the action is repeated and the student reflects upon it, an action that takes place entirely in the mind, is internal, it may be interiorised as the process. With the process conception of a function, an individual can link two or more processes to construct a composition, or reverse the process to obtain inverse functions. It is expected that students attempting the chain rule should at least also work at this level.
- Objects: The student here can reflect on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations can act on that totality and is able to construct such transformations. We say that the process has been encapsulated to an object.
- Schema: A collection of cognitive objects and internal processes for manipulating these objects. Schemas themselves can be treated as objects and included in the organization of higher order schemas, they are said to be thermatized (Cooley, L.; Trigueros, M. & Baker,B. 2007). Asiala (2004) asserts that an individual's schema is the totality of knowledge which for him/her is connected consciously or unconsciously to a particular mathematical topic. For example an individual may have a function schema, derivative schema or chain rule schema.

METHODOLOGY

Qualitative methods were employed and data was collected via a questionnaire administered to a group of calculus students' (n = 23). The questionnaire was designed to give an insight into their knowledge of and skill with functions, composition of functions, differentiation and the chain rule. The data were analyzed to investigate how their performance on the composition of functions items related to that of the chain rule. Follow-up interviews based on some questionnaire responses were conducted with ten subjects. The second part of the pilot study involved interviews with some participants who answered the questionnaire. This was done to describe how those students constructed the concept of the chain rule. The ten participants in the interviews were chosen based on their scores in Part 1, covering a range of chain rule scores and a range of overall scores. The interview followed a structure designed to elicit the student's understanding of the rule based on tasks from the previous instrument.

ANALYSIS AND DISCUSSION OF ITEMS FROM QUESTIONNAIRE

An activity worksheet consisting of twelve items was administered to the 23 students. The items addressed the following skills in the given sequence: (a) Items 1and 2 focused on whether a given graph represented a function or not. (b) Items 3 and 4 focused on the understanding of composition of functions. (c) Items 5.1 to 5.6 dealt with students' applications of rules for derivatives, including the chain rule, and (d) Items 6.1and 6.2 focused on integration where the chain rule is embedded in the structure of the integrand. The 12 items were coded (scored) using a 5 point rubric based on the following guidelines adapted and modified from Carlson (1998).

Score	Description of mental action	Behaviours
5	Made all the mental constructions proposed in the genetic decomposition regarding the concept tested	A complete response to all aspects of the item and indicating complete mathematical understanding of the concept
4	Made most of the bits and pieces of mental constructions of the concept	A partially complete response with minor computational errors, demonstrating understanding of the main idea of the problem
3	Displaying few mental constructions of the concept, with some explanations	Not totally complete in response to all aspects of the item and incomplete reasoning.
2	Displaying few mental constructions with no explanations	No reasoning to justify written response
1	No mental constructions of concept	No written response

	shown at all	

Table 1: Scoring codes used

These guidelines were used to construct specific rubrics for each item. The analysis of the results was based on two considerations: (a) An initial genetic decomposition of the concept of the chain rule was used to guide the researcher's teaching instruction in class and guided the construction of the interview tasks used (b) Piaget's (1985), triad mechanism which consists of three stages. These are referred to as Intra-; Inter- and Trans- which display the development of connections an individual can make between particular constructs within the schema, as well as the coherence of these constructions, Dubinsky (1991). The Intra stage focuses on a single object followed by Inter which is a study of transformations between objects and Trans noted as a schema development connecting actions, processes and objects.

Learners' written work served as a critical source of validation for the questionnaire. By analyzing what the learners wrote, the researcher got an understanding of how students negotiated the mathematics embedded in the context. Figure 2 gives the summary of the scores gained by the participants in each category based on the above description. From the graph below the scores revealed a lower mean score on answers displayed for composition of functions and the higher mean score for the derivative category. This indicates that most students presented correct answers for the latter category even though they did not clearly understand the composition and decomposition of functions.

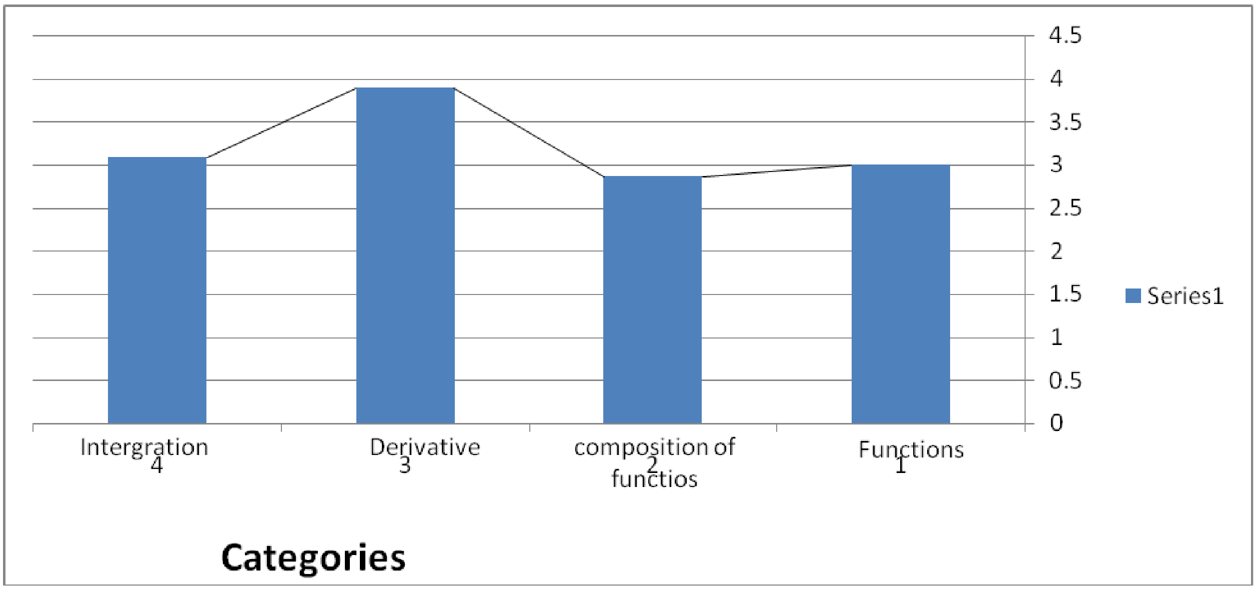

Figure 2: Bar graph displaying mean scores for each category

Category A: Functions

This category had two items focusing on whether a given graph indicated a function or not. The scoring codes indicated in Table 1 were adapted for this category, as indicated in Table 2.

Score	1	2	3	4	5
Indicator	Yes, with incorrect or no response	Yes, but shows some understanding of function concept	No, with incorrect explanation	No, with use of vertical line test, without elaboration	No, with a correct explanation

Table 2: Category A Allocation of scores

> **Item 1**
>
> Is a student correct to identify the following as a function? Explain.

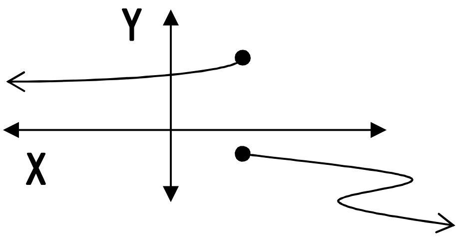

Figure 3: Graph for item 1

The item was designed to test their understanding of the concept of functions represented graphically. It was of interest to know if they would mention the vertical test line with elaboration, continuity of the functions or show misunderstanding of the closed circles in their explanations.

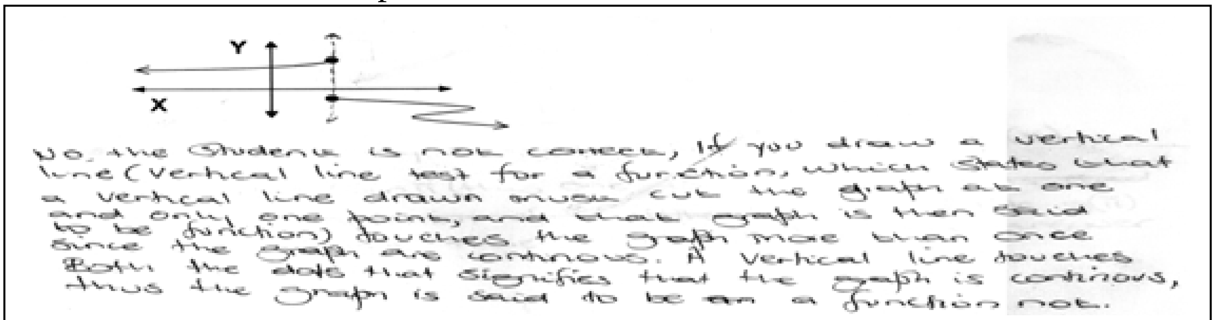

Figure 4: Written response of S 17

The results showed that 13 students out of 23 displayed a clear understanding of graphical representation of a function. They gave a complete response to all aspects of the item indicating a good mathematical understanding of functions represented graphically. For example the response of S17, who was given a score of 5, is shown in Figure 4. In an interview with S4, she indicated that if a vertical test line was done, it would touch the graph more than once on the zigzag part of the graph. Questioned further:

Interviewer: 'Would it make any difference if one of the dots was opened'.

S4: 'To me no difference, I don't really understand what the dots mean'.

In the revised questionnaire Item 1 will be revised and a diagram with the closed dots will be presented without the zigzag part of the graph. This will ensure understanding of the vertical line test and the understanding of continuity of the function.

Item 2

A given correspondence associates 3 with each positive integer, -3 with each negative integer and 1 with 0. A student has marked afore - mentioned relationship as a function, is that correct, support your answer.

This item focused on the students' understanding of the concept of a function represented as a set of ordered pairs. It was important to know whether students would associate elements in the domain with those in the range using mapping and interpreting. This would indicate whether they are the process, object, or schema stage of APOS. Only 5 out of 23 students displayed understanding of the function concept for this item. Interestingly all 5 students gave full explanations for item1. This suggests that they have adequately connected schema which incorporated vertical line test, continuity and mappings. Item 2 is a very suitable item and will be included without changes in the main instrument. It serves a requirement of understanding of functions as prescribed in the initial genetic decomposition.

Category B: **Composition of functions**

This category consisted of two items focusing on composition of functions. This category is included so as to establish whether the students had a process or object understanding with regards to composition of functions. Item 3 dealt with composition and of functions using the '$\circ$' notation.

Item3

Given two functions, $f(x)$ and $g(x)$ such that, $f(x) = 4e^x$, $g(x) = 3\sin x$.

Find $(f \circ g)(\ )$

The scores were allocated as indicated in Table 3.

Score	1	2	3	4	5
Indicator	evaluating $f(\ )$ and $g(\ )$,	subtraction of	for finding $(g \circ f)(\ )$	for arriving at	a completely

	and stopped' no further computation	functions and then evaluating at		$f(g(\ \ )) = 4e^{3\sin}$	correct response

Table 3: Item 3 Allocation of scores

The results showed that most students experienced difficulty in dealing with the composition of functions (only 8 out of 23 displayed complete understanding of composition of functions). It was noticed that the few students who showed evidence of working with composite functions at the object stage of APOS were operating in the action stage when they had to decompose the functions. They were unable to identify individual functions that made the composite function.

Item 4

Given that $(f \circ g)(x) = -10\sin 5x \cos 5x$

4.1 Find functions f and g that satisfy this condition.

4.2 Is there more than one answer to part 4.1? Explain.

This item required decomposition of a composite function and was scored as in Table 4.

Score	1	2	3	4	5
Indicator	left blank	for correct expressions unlabeled and 1 for 4	for "No" for part 4.2; reversed labels (process); or incorrect answer for 4.1 and "Yes" for 4.2	misunderstanding the '$\circ$'notation but indicated understanding of composition	for two correctly labelled pairs for 4.1 and 4.2

Table 4: Item 4 Allocation of scores

Only 3 out of 23 students found correct functions for 4.1 and displayed complete

understanding of decomposition of the given composite function. With the process conception of a function, an individual can link two or more processes to construct a composition, or reverse the process to obtain the original functions. It is expected that students attempting the chain rule at least work at this level. It was found that 15 students operated in the action stage of APOS regarding decomposing the given function. They did not know which steps to take because they were restricted to the formula interpretation of the composed function. They were unable to come up with two or more functions to reverse the given composed function. This item though, will be included in the main instrument for the study without changes.

Category C: Derivative

This category consisted of six items dealing with differentiation of functions.

Item 5.1 to 5.6

Differentiate the following, with respect to x:

5.1 $y = -3\sin x + 2e^{\cos x} - 5e^{x}$

5.2 $y = \cos(2x-5)^3$

5.3 $f(x) = \sin^3(4x)$

5.4 $y = \sin^2(4x^2 + e^{\sqrt{2x-\cos e}})$

5.5 $f(x) = \ln[\cos(7x)]$

5.6 $y = (\cos ec^3 x + e^{\tan x})^2$

For item 5.1 score allocation followed the guide in Table 5.

Score	1	2	3	4	5
Indicator	a guessed answer	not using appropriate rule; or errors with rules	involving elementary differentiation rules	error with derivative of trig functions	correct answer

Table 5: Item 5.1 Allocation of scores

Most students differentiated correctly in this item, (16 out of 23). They mostly operated in the object stage of APOS regarding this item with minor errors of signs occasionally. Items 5.2 to 5.6 dealt with evaluation of the derivative using the chain rule. Scores were allocated as indicated in Table 6.

Score	1	2	3	4	5
Indicator	no evidence of considering chain rule.	applied chain rule indiscriminately; or attempted to avoid chain rule by expanding/rearranging terms (Item 5.2)	If computed mixing composition	a Minor error such as dropping (−) sign or arithmetic errors; or applied	a correct answer

				chain rule but error with derivative rule	

Table 6: Allocation of scores for items 5.2 to 5.6

S16 (Mzi') wrote an explanation of how he got to his answers.

When Mzi was asked what he meant in question 5.2 when he wrote differentiation has to be like peeling an onion, he said:

Mzi: 'To me differentiating such a function that is so loaded with many other functions is like peeling an onion, taking it layer by layer until you get to the inner one.'

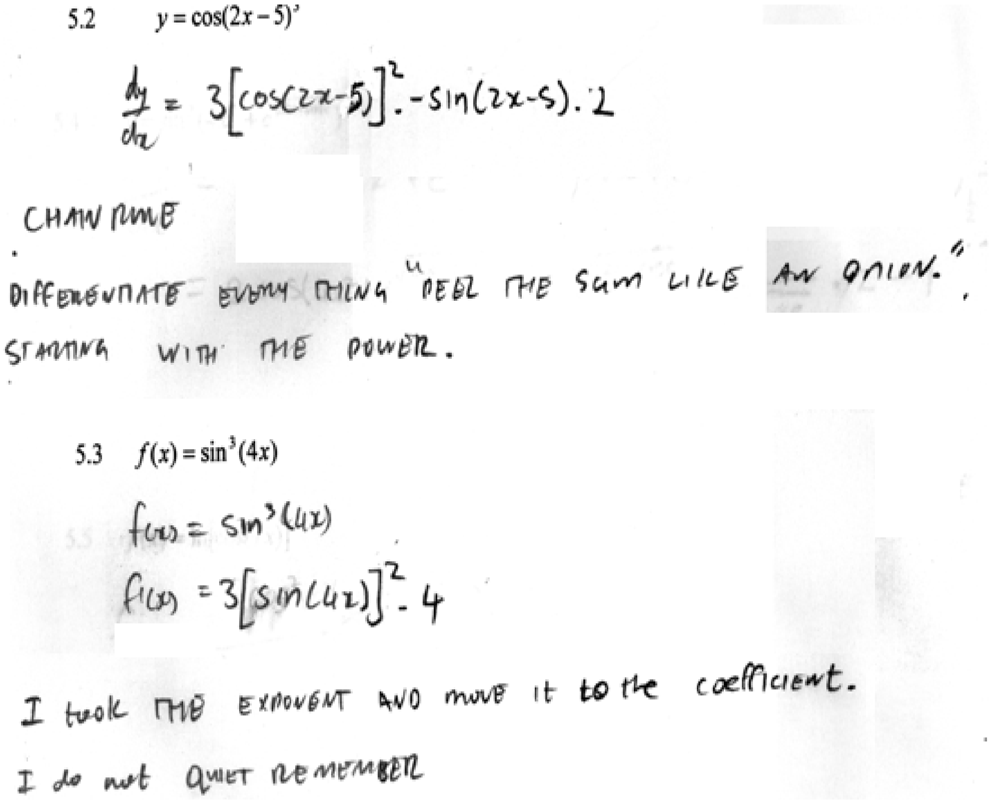

Figure 5: Mzi's written response

Mzi was scored 2 in 5.2 and 3 in 5.3. Figure 5 shows some of Mzi'swritten responses. Indeed in his explanation, one can say he displays a schema of the chain rule. He seems to have learnt and knew the chain rule and was applying it with understanding to different problems. He was (1) able to access the chain rule as per need, (2) able to reflect on it by paying conscious attention to techniques and algorithms used in dealing with chain rule, and (3) understood all the procedures involved in performing calculations involving the chain rule.

Three of the students interviewed acknowledged learning the chain rule, could not express it though but were able to apply it correctly. Also during interviews it was clear that, about 60% of the students except Mzi interchanged function composition with function multiplication.

S13 presented the use of the chain rule as in Figure 7.

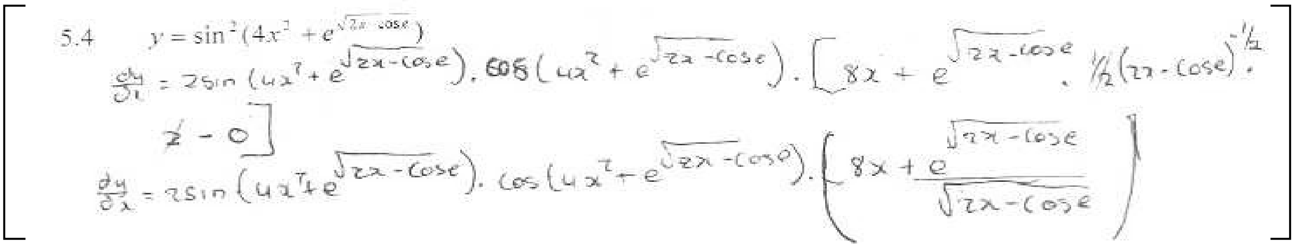

Figure 7: S13 response to item 5.4

Loyiso's (S23) responses on this category were captured as follows:

Interviewer: 'Tell me, or explain how did you come up with your answer in question 5.2?

Loyiso: You see, eh, in $y = \cos(2x-5)^3$, I simply expanded the angle, and multiplied out.

Interviewer: And in 5.3?

Loyiso: I did the same thing, I tried to expand the function, so I split it up as, $f(x) = \sin^2(4x)\sin(4x)$.

Interviewer: And then?

Loyiso: Ngabon' ukuthi ngisebenzise ama identities, lapho u $\sin^2(4x)$ ngimenz'u$1-2\cos(8x)$,ngabese ngiyamultiplaya ngo $\sin 4x$,maqede ke ekugcineni ngathola iderivative. (I thought that I must use the identities where I changed $\sin^2(4x)$ to be $1-2\cos(8x)$, I then multiplied with $\sin 4x$, worked out until I got the derivative.

He left $y = (\cos ec^3 x + e^{\tan x})^2$ after expanding it to be $y = \cos ec^6 x + 2\cos ec^3 x.e^{\tan x} + e^{\tan^2 x}$. One would say that Loyiso has interiorized his actions and objects into a system of operations. He is familiar with a process of differentiation and can carry it out through mental representations. His co-ordination is failing though because he is unable to co-ordinate the composition of functions with derivatives and this is required for the chain rule. According to the triad mechanism of Piaget, he is operating in the inter stage.

Over 60% of students were scored 4 and above in this category. Only 42% of those students used the chain rule in their responses though. Quite a number of students tried to avoid use of the chain rule in their responses. These items will be included without change in the main instrument.

Category D

This category focused on integration where the chain rule is embedded in the structure of the integrand. It consisted of two items.

Items 6.1 and 6.2

Evaluate:

6.1 $\int 2x\sqrt{1+x^2}\,dx$

6.2 $\int (3x+6)dx$

It was important to find out how students would use the chain rule, to find the answer to the given problems. The table of standard integrals was therefore not given to them. Scores were allocated as indicated in Table 7.

Score	1	2	3	4	5
Indicator	incorrect answer	not interpreting the root sign correctly	minor errors of putting incorrect signs	identifying the function, its derivative and then applying standard integrals	for correct answer using chain rule

Table 7: Allocation of scores for category D

For items 6.1 and 6.2, S13 displayed what Piaget refers to as applying an existing schema to a wide range of contexts. He could deal with reversal of the chain rule where the new process required by mentioned questions was constructed.

Students in the inter- stage will show evidence of having collected some or all the differentiation and integration rules in a group and perhaps provide the general statement of the chain rule without yet constructing the underlying structure of the relationships. That student would tackle $\frac{dy}{dx}$ of $y = ((\cos ec^3 x + e^{\tan x})^2)$, by applying the power rule, not sure that he/ she is using the chain rule. This student during interviews, and further questioning would explain the connection between his general statement of the chain rule and its applicability.

CONCLUSIONS AND IMPLICATIONS

It was evident from the results displayed and interviews that followed that even though students wrote down the correct answer, they were not always thinking of the correct answer. There were a few misunderstandings hidden in the notation in function composition and the use of the '$\circ$' notation. To have a schema of the chain rule one has to master the use of differentiation and brackets used in function composition. This involves finding the derivative of each function and multiplying the results as illustrated in the definition of the chain rule. These are students who would be skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consists of a list of unconnected actions, processes and objects to produce correct answers.

There was no need for major revisions to the questionnaire, and this instrument will now be included in the main study. The responses also provided evidence that coincided with the proposed genetic decomposition of the chain rule. The interviews also triangulated the written responses to the questionnaire.

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LEARNERS' UNDERSTANDING OF ZERO: THE NOTHING THAT IS ACTUALLY SOMETHING

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The understanding and application of the concept of zero as a number and placeholder have been a contentious issue over centuries in different cultures. Indian mathematicians invented zero as a placeholder and number in relation to arithmetical operations about 1500 years ago after Babylonian, Roman and Greek mathematicians struggled with and sidelined the idea of 'nothing' representing something. Although the invention of zero may be regarded as the most noteworthy achievement in the history of number the development of the concept of zero is vague in our mathematics curriculum. In this MEd research study, I investigate comprehension of the concept of zero among primary school learners in grades 3 to 6. I also examine understandings of the concept of zero among teachers on BEd and ACE accredited mathematics courses. The study aims to establish why learners and teachers experience problems with the concept of zero. In this paper, I report on the conceptions that grades 3 to 6 learners experience concerning multiplication and division by zero.

MOTIVATION

The idea of investigating primary school learners' problems concerning the concept of zero occurred to me during my involvement as fieldworker in a school-based mathematics development project in the Western Cape during 2004. Observations of Grade 4 learners' conceptions of basic calculations revealed that learners developed misconceptions[30] regarding subtraction involving zero as a digit or place holder in numbers. They often applied an algorithm supplied by the teacher to explain their understanding. During this period, while I was conducting a classroom observation session, a Grade 4 teacher confidently informed the learners that "Zero is not a number"! I seriously considered engaging in formal research on the topic when I co-facilitated a Grade 7 mathematics lesson with a school psychologist who conducted

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[30] The term *conception* in this paper refers to the learner's original or existing understanding of a concept through his or her own experiences or the influence of society. Newly constructed concepts are assimilated and accommodated into the existing conception. If the existing conceptions are correct or incorrect, the new concept might be understood or misrepresented so that it develops into a misconception – not because the learner does not understand, but rather because the learner understands something else (Davis, 1983; Wilcox, 2008). In this study, the use of the term *misconception* is used to refer to misrepresentations. Gerald (2006) described misconceptions as often intelligent generalisations.

mental calculation speed tests to assess learners' basic mental calculation skills in 2007. I was alarmed by the majority of grade 7 learners' inability to solve the calculations 4×0 and $0 \div 7$. This phenomenon strengthened my assumption that the teaching and learning of the concept of zero is problematic in primary school.

The concept of zero is explicitly mentioned for the first time in the current primary school mathematics curriculum in the Grade 5 Assessment Standards. The standards state that learners should be able to "Recognise and represent numbers to describe and compare them: 0 in terms of additive inverses" (South Africa. Department of Education [DoE], 2002:41). Learners in the lower grades are expected to "know number names and read symbols *from* [italics added] 1 to . . ." and "count forwards and backwards" in different intervals "*between* [italics added] 0 and . . ." (South Africa. DoE, 2002:20). The assessment standards obviously do not require of teachers to develop learners' concept of zero, especially with regard to acknowledging zero as an important number in the number system, its role in the place value system and how it behaves in calculations.

Knowledge of the concept of zero is significant in different areas of mathematics. For example, counting and calculating on a number line, reading temperature on a thermometer, using a measuring tape or scale for measuring length, mass, weight and capacity to perform accurate measurements, understanding rational numbers (decimals, e.g. 0,5) and integers, estimation and rounding off, construction of sine, cosine and tan graphs, zero raised to a power, etc. would not be possible without zero as a number and placeholder. People often relate the value of zero to nothing. When counting back and extending the whole numbers to include negative numbers the numbers are expressed as, for example three, two, one, zero (not nothing), minus one, minus two, etc. When reading temperature we do not refer to 'nothing degrees' as the freezing point for water but rather to 'zero degrees'. A decimal number, e.g. 0,5 is read as 'zero comma five or nought comma five' (not nothing comma five). Zero as a digit in secret banking pin numbers or telephone numbers is not referred to as nothing. Although we use the symbol zero continually in mathematical activities learners' persist in misrepresenting calculations concerning basic operations involving zero. This is often the result of learners' tendency to connect the value of zero to nothing. Relating zero to nothing could cloud learners' conception of multiplication and division with zero as reflected in the quote below.

> You can never multiply or divide by zero because zero is nothing. Each and every number you multiply or divide by zero you get zero. And when you subtract one from zero or zero from one you get one. Because when you subtract something from nothing you get the same number, that number that you subtract. Zero is nothing at all. (A Grade 11 learner: additional data produced in this study).

THEORETICAL FRAMEWORK

Solving calculations with zero typically involve shallow rules and procedures disconnected from conceptual understanding. Incoherent conceptual and procedural knowledge is a primary hindrance to mathematics achievement for learners at all levels of development (Semenza, Granà & Girelli, 2006). Evidence produced in this study showed that teaching and learning of abstract concepts are based on algorithms supplied by teachers which could be related to the behaviourist theory of stimulus and response as echoed by Orton (2004). The author claimed that some teaching practices associated with behaviourist learning theories are used in teaching elementary arithmetic. The demonstration of formal ways to solve problems entails a traditional rather than an inquiry-based teaching and learning practice. In the traditional approach teachers assume that learners do not have cognitive tools and strategies of their own to solve problems. Interaction in the traditional teaching sense is based on learner responses initiated and evaluated by the teacher (Mehan, Sinclair & Coulthard in Wood, Cobb & Yackel, 1993).

The theory of constructivism is concerned with cognition, the progression and development of thinking and reasoning as a human action by individuals and between individuals and society. The constructivist theories of Piaget and Vygotsky both advocated and exemplified the “transactional, relational and contextualized” approaches for considering human development through interaction with the environment (Vianna & Stetsenko, 2006:84). For the purpose of this study I adopt the constructivist position of Wood, et al. (1993). The authors maintained that social interaction between the teacher and learners is essential in creating opportunities for learning. Learners should genuinely communicate mathematical thinking and reasoning while involved in problem solving and investigative learning experiences. In addition, I consider constructivists theories that advocate learners’ abilities to draw from their own bank of facts, strategies and principles. Learners apply these abilities in the conceptualization of new ideas by connecting existing and novel ideas (Ginsburg, 1997; Davis, 1983; Olivier, 1989; Kilpatrick, Swafford & Findell, 2001). I am of the opinion that learners construct mathematical concepts through their own experiences and prior understanding. These constructed concepts should concur with the meaning that others construct of the concepts. This process should occur through effective discussion and interaction in order to address learner constructions that develop into misconceptions.

LITERATURE REVIEW

The inclusion of zero into the decimal number system could be regarded as the most remarkable accomplishment in the history of number (Anthony & Walshaw, 2004). Centuries ago, even some of the most influential mathematicians did not use zero as a symbol. Ancient Greeks were brilliant mathematicians but strongly disliked the idea of zero. Archimedes totally disregarded zero and the philosopher Aristotle wanted to have zero banned because it caused confusion when he tried to divide by it. (O’Connor & Robertson, 2000; Ball, 2005). The view exists that the origin of zero is vague but mathematicians are of the opinion that their Hindu colleagues used zero as a number in its own right for the first time. A dot called *sunya* meaning emptiness was used to fill empty columns. The tenth number symbol, zero was thus invented but the number zero was not discovered yet. By 600 AD Indian mathematicians invented zero as we know and use it in calculations today. One hundred and fifty years passed before the Arabs accepted the Indian mathematicians' breakthrough and included zero in their system (Anthony & Walshaw, 2004; Reid, 1956).

The "historical delay" of zero's acceptance as a number is evident of the fundamental complexities of the manipulation of a "null quantity" in mathematics (Semenza, et al., 2006:1110). The complex nature of operations with a zero quantity is reflected in the history of mathematics (Quinn, Lamberg & Perrin, 2008:72). The use of zero in calculations appeared in the books of the Indian mathematicians Brahmagupta, Mahãvira and Bhaskara (O'Connor & Robertson, 2000). Brahmagupta, who created arithmetical rules for operating with zero in the four basic operations, asserted that any number multiplied by zero results in zero but struggled with an explanation for dividing by zero. O'Connor & Robertson (2000) acknowledged Brahmagupta's attempt to define operations with zero as brilliant because he was the first person who attempted the extension of numbers and operations to include negative numbers and zero. Mahãvira updated Brahmagupta's book 200 years later in 830 AD and asserted that a number multiplied by zero is zero. Mahãvira's attempt to define division by zero was however incorrect. He claimed that a number divided by zero results in the same number. Kaplan (1999) argued that the Indian mathematicians were concerned with principles of mathematics but not with proving them.

The whole numbers are considered in their interaction with the four basic operations of arithmetic. To regard zero as nothing implies that zero is disregarded as a number. Zero is not regarded as one of the natural numbers although it is logically and naturally fitting in with these numbers and replies in a similar way as the counting numbers to the question, "How many . . . ?" (Reid, 1956:11). The importance of zero in calculations such as $6 - 6 = 0$ and $0 \times 6 = 0$, e.g. should be considered to allow learners the realisation that zero is a legitimate solution. To be classified as a number, numerals should associate and combine with already existing numbers. For zero to be assigned equivalent grading as the existing numerals, its behaviour in the four basic operations should be understood; Indian mathematicians first did this around 773 AD (Reid, 1956:6).

Rule-based teaching and learning do not facilitate conceptual understanding of multiplication by zero. Although learners could provide accurate responses in mental and written tasks concerning multiplication by zero, they are often not able to illustrate conceptual understanding of multiplication by zero and frequently provide algorithms that they do not make sense of to explain understanding as indicated in the study of Levenson, Tirosh, & Tsamir (2004). The authors asserted that most Grade 5 and 6 learners in their study were able to solve multiplication with zero problems effectively. They claimed that this was an indication that most learners who learnt multiplication in class knew that multiplication by zero results in zero. Those who knew that $3 \times 0 = 0$ and $0 \times 3 = 0$ did not necessarily demonstrate conceptual understanding of multiplication by zero. The authors assigned this occurrence to the fact that learners' original learning experiences involve the natural numbers. Davis (1983) maintained that learners intuitively apply the concept of repeated addition to multiplication, for example $3 \times 5 = 5 + 5 + 5$, but they do not assimilate and accommodate 3×0 (three zero's) and 0×3 (zero three's) in their intuitive structures for multiplication. The choice of structure they impose on 3×0 and 0×3 is mediated by the original structure they impose on multiplication which is repeated addition. Concept development of multiplication should therefore be expanded to alternative structures for multiplication so that learners are aware of the possibility of the application of inaccurate choices.

Van den Heuvel-Panhuizen (2001) and Levenson, Tsamir & Tirosh (2007) reported on learners' problems with division by zero. Learners often struggle with the idea of "dividing nothing into something or something into nothing" (Levenson, et.al, 2007:85). Everyday discourse generates a trivial configuration of zero and causes confusion, which successfully averts a profound concept of the understanding of zero. Learners of all ages find division by zero confusing and therefore require teachers with good quality conceptual insight to assist them in developing understanding of the concept. The teaching of division by zero is normally circumvented in primary school. High school learners who are aware of the fact that division by zero is impossible or undefined are often unable to justify this concept and hold teachers responsible for conveying the fact without explanation (Henry and Reys in Quinn, et al., 2008).

Olivier (1989) indicated that highly effective teaching of zero's role in multiplication could result in even competent learners' misunderstanding of the concept. Sewell (2002) stated that learners begin school with an existing reservoir of knowledge entailing their own elaborations of how they experience and make sense of the world as acquired from personal experiences, social events, the media, people and places. The real world conceptions that learners have of some mathematical concepts could often be in disagreement with scientific views of the subject matter and could be referred to as misconceptions or inaccurate beliefs. Newly constructed concepts are assimilated and accommodated into existing conceptions. Misconceptions are often caused by over generalization of previously correct knowledge which is extended to new learning where it becomes invalid. If the existing conceptions are incorrect, the new concept might be misrepresented because the learner understands something else or answers a different question. It is the teacher's responsibility to determine what the learners actually understand (Davis, 1983; Ginsburg, 1997; Sewell, 2002). Wolsey in Sewell (2002:24) provided an account of the persistent nature of misconceptions when he cautioned almost five centuries ago that one should "Be very, very careful what you put into that head, because you will never, ever get it out".

The literature review in this study primarily highlights that regarding zero as nothing, a lack of conceptual and procedural understanding[31], rule-based learning and the application of ineffective cognitive structures lead to misconceptions concerning the concept of zero. Various instances of learners', teachers' and ancient mathematicians' confusion, uncertainty and inability to justify and explain conceptual understanding of the concept of zero are emphasized. The literature predominantly accentuates the development of misconceptions, which are mainly attributed to the application of isolated, unacquainted rules and procedures and insufficient knowledge of zero as a number and its behaviour in calculations, which hamper effective meaning construction. Effective practical models and constructive discussion and debate could result in positive reasoning and construction of conceptual and procedural understanding of the concept of zero.

METHODOLOGY

This study is performed in an interpretive orientation with the view that each individual has his/her own reality (Von Glaserfeld in Brown, 2009) and constructs his/her own meaning of phenomena in the social world. The research is performed in the qualitative orientation with some quantitative elements in the analysis and discussion of data. The main concern was sense-making of the personal world of the experiences of human beings (Cohen & Manion, 2000) by observing learners' (and teachers') interactions, perceptions and expressions in constructing meaning in their own, authentic learning environments in order to provide detailed descriptions of their actions. The theory of constructivism underpins the study with the assumptions that learners construct meaning in socio-historical contexts and assimilate new meaning into their existing knowledge structures.

The research design is based on ethnography involving a multiple case study with an opportunity sample approach. The production of data involved more than one specific technique and justifications for the use of the techniques concerning the type of understanding being sought (Gough, 2001). Case studies entail single units or multiple individual units in which multiple variables are rigorously examined by interacting with the context/s of the case/s to understand events, actions and processes. The design principles (also applying to other forms of qualitative research) in case studies focus on conceptualization, contextual features, multiple data sources and analytical strategies (Yin & Stake in Babbie, et al., 2001). The intension was to

[31] Conceptual understanding refers to knowledge construction entailing different but related chunks of mathematical knowledge, skills, values and beliefs. This knowledge could be retrieved, assimilated, accommodated, reconstructed and applied in the development of various new concepts to demonstrate understanding without depending on the recall of memorized facts that do not make sense. Learners who have developed conceptual understanding normally know when calculations are wrong or about to go wrong. Procedural understanding involves procedures or steps for solving problems that could be learnt without conceptual understanding or sense-making of the problems and solutions. Effective procedural understanding however entails the application of efficient steps to calculate easier and smarter to illustrate conceptual understanding (Kilpatrick, Swafford & Findell, 2001).

identify emergent patterns from which I was able to produce generalizations (Connole, 1998; Babbie, et al., 2001; Danermark, et al., 2002).

The study was conducted in three stages from 2007 involving six cases which included teachers and learners in the Western and Southern Cape and the Eastern Cape in South Africa. Stage 1 involved Grade 5 and 6 and Grade 3 and 4 multi-grade classes. In Stage 2, the focus was on BEd and ACE teachers and Grade 5 learners were involved in Stage 3. For the purpose of this paper, the focus is on data production in the grade 3 to 6 classrooms that occurred during Stages 1 and 3. A mental calculation questionnaire requiring instant responses to multiplication and division problems was completed by grades 3 to 6 learners. Responses to the problems 4 × 0 and 0 ÷ 7 were analyzed and discussed. I developed an additional questionnaire requiring written elaborations of learners' conceptualisation based on the study of Van der Heuvel-Panhuizen (2001). I included the tasks 1 – 0 = ☐, 0 – 1 = ☐, 1 × 0 = ☐, 0 × 1 = ☐, 0 ÷ 1 = ☐ and 1 ÷ 0 = ☐. The Grade 3 and 4 multi-grade class was engaged in a lesson to develop understanding of multiplication by zero because most of the learners solved the problem 4 × 0 incorrectly in the mental questionnaire. The learners were requested to work in groups and to make drawings of their understanding of multiplication problems with natural numbers and then with zero. I recorded tasks such as 4 × 1 = ☐; 4 × 4 = ☐; 4 × 3 = ☐ and 4 × 2 = ☐ on the writing board. The number sentences were recorded randomly to prevent learners from solving the problems through pattern recognition but rather by thinking and reasoning authentically to show the cognitive structures they imposed on the problems. The problem 4 × 0 = ☐ was included after the learners solved the problems involving only natural numbers. This activity was followed by an intervention to assist learners in developing understanding of multiplication by zero through mediation. The learners were also engaged in physical modeling of multiplication problems to demonstrate understanding. I photographed this learning and teaching process. For the Grade 5 semi-structured focus group interview, I selected three learners. Because learners worked collaboratively in the written elaboration task, I was interested in individual learners' understanding of multiplication and division by zero.

DISCUSSION OF RESULTS

Multiplication by zero

The study provided evidence that accentuate my assumption that Grade 3 to 6 learners experience problems with multiplication by zero. Correct solutions provided in mental calculation tasks involving zero were not proof of learners' conceptual understanding of multiplication by zero. The intuitive structures that Grade 3 and 4 learners applied to demonstrate understanding of multiplication by natural numbers did not accommodate conceptualization of multiplication by zero. In the mental calculation speed tests 34% of Grade 3 and 4 correctly responded that 4 × 0 = 0 but none of these learners were able to demonstrate conceptual understanding of the concept. These learners displayed effective understanding of multiplication of single-digit natural numbers when they were requested to draw their understanding of multiplication during the written elaboration classroom activity. They intuitively assimilated counting strategies into repeated addition structures to represent multiplication by natural numbers by illustrating pictorially, e.g. that 3 × 4 = ♦♦♦♦+ ♦♦♦♦+ ♦♦♦♦= 12 or 3 × 1 = ♦ + ♦ + ♦ = 3, thinking about multiplication as, for example ‘three times four’, i.e. a group of four individual objects repeated three times. They also displayed competence in linking the practically based illustrations to symbolic representations or mathematically based statements, e.g. 2 × 3 = 6; 3 × 4 = 12, etc. The learners however struggled when they had to illustrate understanding of, e.g. 2 × 0 as seen in Table 2 below. One group claimed that they could not draw zero. The structure used in the multiplication with natural numbers tasks did not suffice the operation with zero. In their attempts to solve 2 × 0, some learners recorded correctly that *2 times 0 = 0* but they experienced cognitive conflict when their intuitive pictorial structures could not accommodate multiplication by zero.

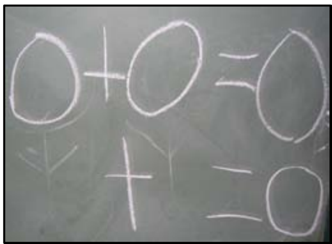	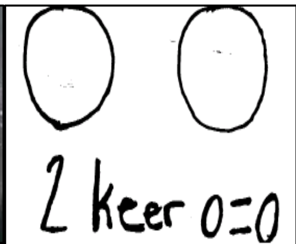	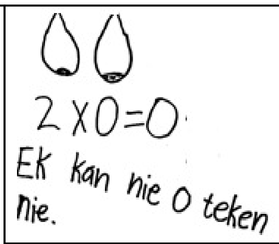	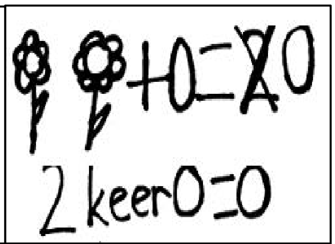
	2 times 0 = 0	I can’t draw zero	2 times 0 = 0

Table 2: Grade 3 and 4 learners’ struggle to represent multiplication by zero

An average of 70% of Grade 5 and 6 learners responded that 4 × 0 = 0 in the mental calculation task. In the written elaboration tasks all of the Grade 5 and 6 learners involved in Stage 1 of the study knew that 0 × 1 = 0 and 1 × 0 = 0 but only 25% of the learners displayed efficient multiplicative thinking through the concept of grouping and composite wholes, e.g. “If you multiply nothing once, it stays nothing”, “I have no groups of one then I have nothing” or “I have one group of zero. I have nothing”. The rest of these learners (75%) were not able to provide sensible justifications for their solutions. They over generalized or exploited the concept of subtraction in elaborations using terminology such as ‘lost it’, ‘ate it’ and ‘give away’ and thus provided flawed explanations to explain the action of multiplication by zero, e.g. “I have nothing. I got one sweet and lost it”. This learner reasoned that 0 + 1 – 1 = 0, which is true but different from 0 × 1 = 0, which are zero groups of one. Grade 5 learners in the focus group interview applied the concept of doubling to the calculation 1 × 0 = 0, i.e. “You double zero by one and you still get zero” and “You can’t double nothing”. This shows that, if learners are not able to assimilate new knowledge into existing knowledge they invent and over generalize cognitive structures which are disconnected from to the problem without consideration of the mathematical implications.

The Grade 5 learners (25%) in Stage 3 who responded that 0 × 1 = 1, provided rule-based explanations, for example "If you have 0 × 1 the answer will be 1". Some learners (50%) who claimed that 1 × 0 = 1 asserted that multiplying by zero is impossible because ". . . you can't times/× zero by a big number" probably alluding to the misconception that you cannot subtract a big number from zero, e.g. 50 – 18.

One of the Grade 5 learners in the focus group interview demonstrated understanding of commutativity when he recognized that 3 × 1 = 1 × 3 because the numbers are "reversed", i.e. "It's still the same thing. You just reversed it. You can still say one plus one plus one". The learner claimed that both expressions mean 1 + 1 + 1 while 1 × 3 means 1 group of 3. This learner did however not apply the commutative property to multiplication with zero and claimed that 1 × 0 = 1 and 0 × 1 = 0 which is an indication of learners' tendency to separate zero from the natural numbers. The learner knew that 3 × 1 = 1 × 3 but did not realize that 1 × 0 = 0 × 1.

Knowing the rule for multiplication by zero does not guarantee conceptual understanding of the process (Olivier, 1989). Some of the Grade 6 learners found it difficult to represent their thinking processes relating to multiplication by zero although they knew multiplication by zero results in zero. These learners asserted that multiplication by zero is easier said than done; they did not make sense of it and found it difficult to explain:

> I took the sum zero multiplied by one. It looked very easy to do but it was not as easy as I thought. I feel it is a bit difficult. I never understood it. It is also difficult to explain. But I tried to explain. At the beginning it was easy to write down the answer. But I struggled to write down the explanation.

Some of the Grade 5 and 6 learners (25%) supplied rule-based explanations displaying procedural but not conceptual understanding, e.g. "If you take 1 × 0 the answer will be 0". One learner blamed his Grade 3 teacher for providing the rule that zero multiplied by any number is equal to zero but not explaining why this was the case.

> I think in grade three then the teacher taught us about zero times but she did not explain why. So we don't really, really know why. But I know zero times one is zero but we don't know the explanation . . . She's just told us zero times one is zero or if you can say zero times one hundred it still remain zero, but she did not explain clearly that. Like when I said, teacher can you please explain that sum to me but she did not explain it.

The majority of Grade 5 and 6 learners in this study did not display previous knowledge of multiplication to impose on multiplication by zero. It appeared that they mostly provided explanations for different problems than the ones they were supposed to solve and explain as suggested by Davis, 1983; Ginsburg, 1997; Sewell, 2002. Learners' elaborations on multiplication tasks involving zero were flawed by their persistent referral to the concepts of subtraction, addition and doubling which hampered the construction of effective mathematical structure. Grade 3 and 4 learners were not capable of constructing an alternative cognitive structure to accommodate conceptualisation of multiplication by zero. The structure of repeated addition involving sets with individual objects that were not combined into single sets developed cognitive conflict and hindered conceptualisation of multiplication by zero.

Division by zero as the dividend (0 ÷ 1)

All the Grade 5 and 6 learners in Stage 1 of the study solved the problem 0 ÷ 1 = 0 successfully and 43% of these learners were able to justify solutions effectively by referring to 'sharing' and 'grouping'. These learners used everyday discourse and multiplicative structures to justify solutions by declaring that "I have no apples. I share it with one friend and nothing is left. I have nothing to share. I cannot give anybody anything" and "There are no groups of one in zero so the answer stays zero". Most of the Grade 5 learners in Stage 3 argued that 0 ÷ 1 = 0 and 50% provided reasonable validations for their reasoning. Some groups maintained that division would not be performed at all while others implied that no-one would get anything; there would be zero items left or if there is nobody to share with the answer is zero, e.g. "There is nobody, one ball and no one to play with it so its equal to zero" and "If I have no banana's so then I can't devide anything to anyone".

Grade 5 and 6 learners in Stage 1 (28%) applied the concept of subtraction to illustrate division by asserting, e.g. "Mother buys me one pencil. I lost it now I have nothing" and "I have one rand and I lost it. Now I have nothing" (implying that 1 – 1 = 0). The Stage 3 learners (38%) applied subtraction to describe division by zero by referring to 'give away', e.g. "If you have 4 chocolet you give away 4 chocolet to 4 children then you have 0 left" (implying that 4 – 4 = 0 or 4 ÷ 4 = 0).

Grade 5 and 6 learners who knew that 0 ÷ 1 = 0 could not justify the solution and mentioned that they did not make sense of the rule applied to this concept:

> The reason why I cannot solve these problems so well is because I do not understand why the answer is zero and I am not good at explaining mathematical terms. I know how I got the answer but to explain it is a bit of a problem. It is difficult for me to put the explanation of the sum in words. If you divide zero by any number, it stays zero. For example: 0 ÷ 9= 0, 0 ÷ 900 =0, 0 ÷ 9000 = 0.

During the Grade 5 focus group interview I checked the learners' understanding of the concept of division and realized that they defined the concept quite accurately by connecting it to equal sharing while they explained multiplication as e.g. "You times", which did not display conceptual understanding. At the beginning of the interview I placed a container with counters on the table without referring to it. The learners ignored the counters during the discussions on multiplication but spontaneously reached for the counters when they demonstrated understanding of division by zero. They imposed real life situations and applied mathematically based accounts in elaborations and were able to effectively demonstrate that 10 ÷ 5 means 2 + 2 + 2 + 2 + 2 and 17 ÷ 2 = 8 remainder 1 or 8½. One of the learners argued that division means “You share so that people get equal parts”. These learners were able to construct a generalization for division by zero as a dividend:

> It will still remain that zero… because you have nothing to give. If you have hundred people and you have nothing to give. Let’s say they must get their salaries and I don’t have any money. I can’t give them anything, any money because I have no money.

Division by zero as the divisor (1 ÷ 0)

I realized that this was an unfair question to pose to the Grade 5 and 6 learners because I did not expect of them to know that division by zero is undefined. The problem however provided insightful understanding of learners’ conceptualization of this extraordinarily abstract concept. Some learners intuitively knew that dividing by zero is not viable. All Grade 5 and 6 learners in Stage 1 reasoned that 1 ÷ 0 = 0 and 40% of them asserted that, if there is one object but no subject nobody will get anything because nothing can be given away or shared. These learners indirectly implied that no division would or could be performed by asserting that “If I have one apple. I divide it among no friends so there is nothing. I cannot give to anybody” and “If I have one banana and I give nobody anything, then nobody gets anything”. I supposed that the use of the term ‘give’ in the second explanation referred to sharing and not subtraction. Limited understanding or awareness of zero’s behaviour as dividend pressurized these learners to supply a numerical solution, i.e. 1 ÷ 0 = 0, instead of asserting that you cannot divide by zero or that it is senseless to divide by zero as implied by the responses above. Learners in Stage 3 (37%) who asserted that 1 ÷ 0 = 0, linked division to subtraction while some learners indirectly asserted that division by zero is impossible. They reasoned, e.g. that “If I have no bananas so then I can’t devide anything to anyone” but they still insisted that the solution is zero. This could be the result of how mathematics is taught in primary school – that there should be numerical solutions to problems. If learners understand that division by zero is not allowed or impossible, they should be encouraged to assert that, e.g. 1 ÷ 0 cannot be done or even that division by zero is silly!

The majority of Grade 5 learners (63%) in Stage 3 reasoned that 1 ÷ 0 = 1 because there is one object and no-one to share it with so the original object remains undivided or is left intact as depicted below. “Let’s say I have one slice of bread and there’s no one to share it with, it will stay one because no one is going to eat it. That’s my answer”. This reasoning is parallel to that of a Grade 6 learner in the study of Van den Heuvel-Panhuizen (2001) who asserted that:

> . . . I’ve found something illogical in arithmetic. That the problem 1 ÷ 0 has no answer, at least not in grade 6. The teacher said that later you learn that the result is infinity. But I think that’s illogical. I think 1 ÷ 0 = 1. Because if I have a cake and invite people round and no one comes then the cake doesn’t have to be cut up and I still have one cake left . . .

Grade 5 and 6 learners in this study illustrated better understanding of division with zero than multiplication with zero. It appeared that the inherent concepts of sharing and grouping relating to division were more naturally habituated in learners' existing schemas than repeated addition and grouping related to multiplication, probably because equal sharing activities are more often applied in their social environments. This finding is in agreement with research studies reporting that learners might be more efficient in solving division than multiplication problems and that young learners have the ability to solve problems involving fair sharing through the process of partitioning (Davis & Pitkethly in Roberts, 2003; Mulligan & Wright, 2000).

IMPLICATIONS FOR MATHEMATICS TEACHING AND LEARNING

The findings in this study illuminated the difficulties and successes that grade 3 to 6 learners experience with the concept of zero concerning multiplication and division by zero. The development of the concept of zero in teaching and learning mathematics has been neglected in the South African mathematics curriculum although knowledge of the concept is significant in different areas of mathematics. The current mathematics curriculum (South Africa. DoE, 2002) does not require of teachers to teach the concept of zero. Levenson, et.al. (2007) mentioned that the Israel National Mathematics Curriculum (INMC, 2005) guidelines include a note to teachers stating that zero is an even number but teachers need not address the issue unless it is raised by learners in the classroom. The Principles and Standards for School Mathematics (NCTM, 2000) on the other hand included an account of a Grade 1 learner to highlight young learners' competence in mathematical thinking and reasoning when they are required to rationalize inferences. A grade 1 learner asserted that, "If zero were odd, then zero and one would be two odd numbers in a row. But even and odd numbers alternate so zero must be even" supplying an informal proof by contradiction for the argument (Levenson, et.al., 2007). The South African DoE is currently embarking on streamlining the existing curriculum in the form of the Curriculum and Assessment Policy Statement (CAPS). In the draft curriculum document released in 2010 the concept of zero is introduced in grade R (South Africa. DBE, 2010:6).

Contrasting views exist about the inclusion of the concept of zero in the primary school mathematics curriculum. Some researchers claimed that learners will not develop a total positive perception of zero prior to their second year of schooling (Wellman & Miller in Semenza, et.al, 2006). Others were of the opinion that the perception of zero does not develop satisfactorily until learners achieve the stage of formal operations (Inhelder & Piaget, 1969). Oesterle (in Anthony & Walshaw, 2004) maintained that primary school learners should not be engaged in calculations with zero and that there is no actual basis for initiating zero facts in whichever basic operations until addition and multiplication of two-digit numbers are introduced. I do not concur with the suggestions of these researchers. In my opinion young learners should be able to construct meaning of the concept of zero through mediation which entails constructive learning experiences in a constructivist teaching and learning approach. Learners should be granted opportunities to construct their own meaning of zero as the empty set, zero as a whole number, counting number and even number, zero as a place holder and zero's behaviour in the basic operations without algorithms supplied by the teacher. Various researchers advocated the inclusion of zero in early learning experiences. Anthony & Walshaw (2004) asserted that learners will have trouble with the understanding of zero's role in place value notation if they do not have a sound understanding of the character of zero. Cockburn (1999) stated that work on the concept of zero should be included in learners' early learning experiences. The study of Van der Heuvel-Panhuizen (2001) proved that Grade 6 learners are able to construct understanding of division by zero if the learning environment is conducive to constructivist teaching and learning. Wilcox (2008:204) engaged her Grade 1daughter in a game involving a number line confronting her with the perception of the value of zero and the notion of negative numbers. After successful meaning construction of these concepts, the child exclaimed, "Zero is a hero!"

I am in agreement with researchers who asserted that teaching of the concept of zero requires competent teachers because of the abstract nature of the concept. Professional development courses and teacher development programs should focus on both content and pedagogical content development concerning the concept of zero. Courses should assist teachers in developing knowledge for teaching (Shulman, 1986; Ball, 2003; Kahan, Cooper & Bethea, 2003; Ball, Thames & Phelps, 2008). Textbook authors and curriculum developers should become aware of the difficulties that both learners and teachers experience with the conceptualization of zero's qualities as reported by various researchers (Wheeler & Feghali, 1983; Reys & Gouws in Wheeler and Feghali, 1983; Cockburn, 1999; Quinn, et.al., 2008; Wilcox, 2008; etc). Training and development programs should raise awareness of the qualities and uses of the concept of zero because what teachers believe, know and decide have a profound effect on the way they teach as well as on students' learning in their classrooms (Fenema, Carpenter & Loef, 1989).

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SHORT PAPERS

THE COMPOUND INTEREST FORMULA AS A MODEL OF COMPOUND GROWTH

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The compound interest formula can be seen as a model to predict the amount of interest that will accumulate over time. However, in schools the formula is generally taught simply as a mechanism for calculating answers. In order to understand the formula as a model, one first needs to understand how interest is calculated in banks. Then one can see that the compound interest formula is an accurate and efficient model of a tedious process, regardless of the day-count convention that is being adopted.

INTRODUCTION

Financial mathematics is one area of the school mathematics curriculum where meaningful applications of mathematics can potentially be found. However, when one considers how this section is assessed in the Grade 12 national assessments, and what has been included in many text books, one might be forgiven for asking whether learners really see financial mathematics as an application of mathematics in the real world or as a model of what happens in banks. Perhaps they simply see the work they do on simple and compound interest, nominal and effective rates, and annuities as a bunch of formulae and procedures to be learnt to pass an examination.

Mathematical modelling in schools is a growing area of interest in the international mathematics education community. In the South African curriculum documents there has been an emphasis on mathematical modelling (DoE, 2003). However, based on anecdotal evidence and discussions with teachers in a wide range of schools, it seems that this emphasis on modelling may have had limited impact at the level of the classroom. Perhaps it is not surprising that there is far less emphasis on mathematical modelling at the level of the intended curriculum based on the most recent versions of the Curriculum and Assessment Policy Statement (CAPS) (DoE, 2011).

Despite the reduction in emphasis on modelling, in this paper I will argue that unless learners have some understanding of how interest works in the world of banking, they cannot see the work they do on financial mathematics at school as a model of what happens in the real world. I will focus only on the calculation of compound interest and show that when we look at the actual calculations done by banks, the compound interest formula proves to be a very good model of the accumulation of interest in that it is both accurate and efficient.

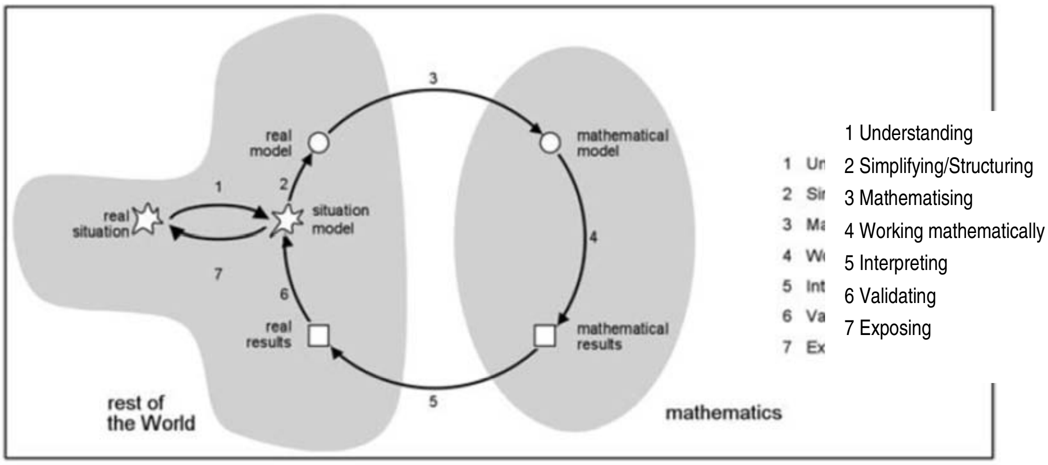
I begin by describing a view of the modelling process that I will use to build my argument. I will then discuss how banks calculate interest, showing the importance of distinguishing between *calculation* of interest and *compounding* of interest. Thereafter I will provide evidence to show that the compound interest formula is a good model.

CONCEPTUALSING THE MODELLING PROCESS IN SCHOOL MATHEMATICS

In the past 20 years various representations of the modelling process have been proposed for school level (e.g. Galbraith and Clatworthy, 1990; Stillman, Galbraith, Brown and Edwards, 2007; Blum and Leiss, 2007), each foregrounding different aspects of the modelling process. In this paper I draw on the model proposed by Blum and Leiss, since it is becoming increasingly widely used in the ICTMA community (e.g. Borromeo-Ferri, 2010; Mass, 2010), and because it is the one that pays most attention to the complexities of initially formulating a model.

Blum and Leiss's (2007) representation of the modelling process (see. fig. 1) distinguishes clearly between "mathematics" and the "rest of the world". As with all the other models, it recognises the iterative nature of the modelling process and reflects the current focus of attention in the modelling community on modelling competencies. This is captured in the attention to the verbs connecting different stages/outputs within the modelling cycle. They also emphasise the transitions between various outputs at different stages of the modelling process, and acknowledge that the actual modelling process is not as linear as their diagram suggests. Space does not permit a discussion of all aspects of the model. I shall focus only on the aspects that are essential for my argument.

Fig. 1 : Blum and Leiss model of the modelling process (from Leiss et al, 2010)



The notion of *situation model* is an important contribution to understanding the modelling process. It can be conceived as a mental model and provides an intermediate step between the real world situation and a model of that situation (the *real model*). The construction of the mental model is an initial step in understanding the real world problem which might involve one or more of the following: observing the real world situation in real time; watching video footage, a simulation or demonstration of the situation; or reading a textual description of the real world situation and the problem to be solved. The situation model probably begins as a synthesis of initial sense-making efforts and may be an accumulation of partial understandings with internal contradictions not yet evident to the learner. From a mathematical point of view, it could also involve many repeated calculations to establish trends and patterns, and develop some sense of key factors in the problem and possible relationships between them – all leading to a deeper understanding of the structure of the problem. The learner's familiarity with the real world situation will impact the extent to which s/he will understand the real world problem, the time taken to construct a situation model and the accuracy of that model. All this is necessary to produce a ***real model***. Unless one understands what the key factors are and how they relate to each other, one cannot begin the analytical work necessary to simplify, to structure and to idealise the situation model to produce a real model (Mass, 2010).

COMPOUND INTEREST

While sophisticated mathematical modelling, research and development are central features of the financial sector, almost no consideration has been given to fundamental aspects such as simple and compound interest as models of real world activity. Similarly there is very little research relating to student conceptions of compound interest although there are a few studies that report struggles to understand and compute compound interest. For example, Beal and Delpachitra (2003) report on a cross-faculty study with over 800 university students in a large Australian university, most of whom were in their first year of study. Students were asked to decide whether a deposit of $100 would earn more interest at 12% p.a. simple interest or at 1% per month compound interest. The choices available were a) $112, b) more than $112, c) less than $112 or d) can't say. Only 52.9% answered correctly and 29% of students thought there was no difference between simple and compound interest.

I now move to discuss the calculation of interest in the banking sector and discuss one unintended consequence of the way interest is dealt with at school. I am well aware that institutional and curriculum constraints in schools make it difficult for teachers to spend additional time on financial mathematics. However, compound interest provides a meaningful opportunity for learners to appreciate the power of mathematics to connect with the real world. Furthermore, I believe that even if curriculum constraints mitigate against this level of detail, it is nevertheless important for teachers to understand the compound interest formula as a model to predict the growth of interest, and not just a tool for calculating answers.

Calculating interest in the real world

In the world of banking, interest is calculated daily and compound monthly. This means that interest calculations done each day use a daily interest rate, and interest is calculated on the balance in the account at midnight. However, the interest is not added to the account until midnight on the last day of the month. Hence the need to distinguish between *calculating* interest and *compounding* interest. I find it useful to explain this using the analogy of a large bucket and a small bucket. The account balance is stored in the large bucket and any transactions during the month impact the balance in this bucket. The small bucket accumulates the interest each day but this is not visible to the client during the month. At month end, the small bucket is emptied into the large bucket. This increases the opening balance for the new month. Interest will then be calculated on the capital and the interest from the previous month – hence compounding of interest monthly. In the new month, interest accumulates again in the small bucket until the end of the month when it will be tipped into the large bucket again. The interest calculated each day is simple interest since it accumulates on the daily balance and does not include interest from previous days in the month. At the start of the following month, the new balance includes interest from the previous month. Thus by definition, interest is being compounded at this point since it is being calculated on the latest balance which includes interest.

Although this reflects a combination of simple and compound interest, banks are not using the simple or compound interest formulae in their daily calculations of interest. They are merely doing a percentage calculation to determine the amount of interest for the day. These daily interest calculations provide the necessary flexibility for banks to deal with transactions at any stage of the month.

The simple and compound interest formulae ($A = P(1+i \cdot n)$ and $A = P(1+i)^n$) are simply models that enable us to make sense of what is happening to a particular sum of money over time. The compounding effect happens as a result of adding the accumulated interest to the account and then calculating interest on this new amount the following month. The formulae enable us to do one calculation that represents (or rather "predicts") the amount of interest accumulated over a certain period.

Introducing the notion of interest in school mathematics

The notion of interest on money is introduced in the South African Mathematics curriculum in Grade 7 (DoE, 2002). In most secondary school texts, the first encounter with interest deals with simple interest, followed by compound interest with annual compounding. Thereafter shorter compounding periods are introduced, but very few text books deal with the reality of daily interest calculations and monthly compounding.

One task that is typical of both local and international texts requires learners to compare the products of two hypothetical banks where one offers simple interest on a lump sum and the other offers a slightly lower rate with compound interest. The fact that such tasks are pervasive in both local and international texts suggests that these may be appropriate ways of introducing learners to the notions of simple and compound interest but this is an empirical question that deserves further investigation. In my experience such tasks tend to emphasise that the simple interest scenario is characterised by equal amounts of interest for each period, rather than interest calculated on the principal amount. It is only a small step then to ***define*** simple interest scenarios as those where the same amount of interest is earned each period. The flaw in this definition is revealed when one considers daily interest calculations. Take for example, a hypothetical notice deposit account with simple interest calculated only on the principal amount each day. The amount of interest that accumulates in May will be more than the interest that accumulates in June because May has one more day than June. So while simple interest is at work, the interest amounts in each month are different. This challenges the "equal interest" definition of simple interest. But if daily interest scenarios are not given attention, the error in the definition may not be exposed. Unless learners are explicitly taught about daily calculation of interest and monthly compounding, such unintended consequences are not likely to be avoided.

MODELLING INTEREST

In this section I continue the discussion of how banks calculate daily interest. I also explain briefly the notion of international day-count conventions. I then compare the actual answers with those one would obtain from the compound interest formula to show empirically that the compound interest formula is an accurate model. Thereafter I link school tasks on compound interest to the modelling cycle proposed by Blum and Leiss (2007).

Daily interest calculations

The daily calculation of interest is a tedious exercise if done manually. In order to predict future values, we need an efficient method that will provide a good prediction under given constraints. Consider the example: R5000 is invested at 12% p.a. for a year with daily calculation of interest and monthly compounding. This scenario can be represented by the spreadsheet (Table 1) which calculates daily interest for each month, assuming 28 days in February. The accumulated interest is added to the account at the end of each month. All calculations have been done with maximum accuracy on the spreadsheet although only some columns show six or more decimal places.

The spreadsheet gives an accumulated amount of R5634.122488 (to 6 decimal places) while the compound interest formula gives an answer of R5634.125151 (to 6 decimal places). The difference is negligible – less than 0.0004%. (I am considering the spreadsheet answer as the ***actual*** answer and the answer from the compound interest formula as the ***predicted*** answer.) In a leap year, the spreadsheet answer would be R635.957907 while the compound interest formula remains unchanged – a difference of less than 0.2882%.

These figures show that the compound interest formula provides accurate answers but it is important to consider a few more details of the real world scenario and the assumptions of the formula. The formula assumes that all months have the same number of days and that there are an equal number of days in each year. From a modelling perspective it is important to recognise these assumptions since they do not reflect the actual numbers of days in different months and years. But how much difference does it make?

With regard to the real world: there are a range of different day-count conventions in use across the world (van Zyl, 2006). Rand-based products use a 365-day convention. This means that the daily interest rate is calculated by dividing the nominal annual rate by 365 irrespective of the actual number of days in the year. In the example above we get 0.12÷365 = 0.00032876... By contrast the typical standard for US dollar-based markets is 360 days, and there are other markets that use an actual-day convention where the daily interest rate is calculated by dividing the nominal annual rate by the actual number of days in the year. These different conventions will impact the daily interest rate and hence the amount of interest that will accumulate.

Month	No of days	Opening balance	Daily interest rate	Daily interest	Cumulative interest for period	Closing balance
Jan	31	5000.00	0.00032876712	1.643836	50.958904	5050.96
Feb	28	5050.96	0.00032876712	1.660589	46.496498	5097.46
Mar	31	5097.46	0.00032876712	1.675876	51.952148	5149.41
Apr	30	5149.41	0.00032876712	1.692956	50.788677	5200.20
May	31	5200.20	0.00032876712	1.709654	52.999260	5253.20
Jun	30	5253.20	0.00032876712	1.727078	51.812339	5305.01
Jul	31	5305.01	0.00032876712	1.744112	54.067477	5359.08
Aug	31	5359.08	0.00032876712	1.761888	54.618521	5413.69
Sep	30	5413.69	0.00032876712	1.779845	53.395336	5467.09
Oct	31	5467.09	0.00032876712	1.797399	55.719374	5522.81
Nov	30	5522.81	0.00032876712	1.815718	54.471536	5577.28
Dec	31	5577.28	0.00032876712	1.833626	56.842416	5634.12
TOTAL	**365**				**634.122488**	

Table 1: Accumulated interest using daily interest calculations and monthly compounding

Table 2 provides a summary of the accuracy of the compound interest formula as a model for these three different day-count conventions. I have chosen the most extreme case for each scenario in order to determine the maximum possible error. The table shows that the formula over-predicts in two cases and under-predicts in the other two. The maximum error for these scenarios is less than 1.75%, and this is an outlier amongst the four cases.

Day-count	Actual number	Actual	Predicted	Difference	% error

convention	of days in year	(from spreadsheet)	(CI formula)	(actual – predicted)	
360	366	645.290428	634.125151	11.165278	1.7303%
365	366	635.957907	634.125151	1.832756	0.2882%
actual	365	634.122488	634.125151	-0.002662	-0.0004%
actual	366	634.123666	634.125151	-0.001485	-0.0002%

Table 2: Comparison of actual and predicted values for daily interest calculations

Up to this point I have focused on full years. If one consider only parts of years, then additional comparisons could be done dealing with the actual months involved. For example there are 181 days in the first six months of a non-leap year and 184 days in the second half of the year. This makes a difference to the actual interest gained when daily interest calculations are considered but there would be no difference in the predicted values from the formula.

Appreciating the compound interest formula as a model

In order to speak of the accuracy and efficiency of the compound interest formula *as a model* we need to understand that banks don't actually use the compound interest formula in daily interest calculations and we also need to know what banks *actually do* to calculate interest. Neither of these issues is given attention in financial mathematics at school. The efficiency and elegance of the compound interest formula goes unappreciated because learners don't do the tedious calculations captured in the tables above – and in banks there would be *one line per day* whereas the tables collapse an entire month into a single line. Because learners don't learn about the basis on which banks work with interest daily, they cannot see the compound interest formula as a model *of something*.

Many text books scaffold the iterative calculation of interest per period and show inductively how the formula emerges as a generalisation of a pattern. At this level, its status is no different to the formulae learners derive for well-known investigative tasks such as the Hand-shake problem, the Frogs investigation, or match-stick patterns that are now common-place in South African mathematics text books. In each of these examples an explicit formula gives the *exact* answer and provides a mechanism to avoid the iterative calculations. While the formula is a model of the structure of the situation, there is no need to simplify the problem and ignore certain details in order to begin the modelling process.

In the case of the compound interest formula, the situation should theoretically be different because it is related to a practice in the real world which, in terms of modelling, we wish to simplify and structure. Therefore the answer provided by the formula may not be the exact answer that would be obtained in a bank because the mathematics that will be done ignores the details of daily interest calculations. But in modelling we are not as concerned with numerical precision as with efficient and sensible approximations. When learners are given tables of values containing principal amounts and interest rates, they are doing the same as they would with tables of numbers of matchsticks in each figure. So what should be a model of a real world problem comes to be an inductive exercise in pattern recognition because, based on the modelling process of Blum and Leiss, the mathematisation has been done for learners and they are working only in the mathematical world. We could argue that learners are interpreting their answer by explaining its meaning as the accumulated amount over time but they cannot validate their answers unless they can compare them against "the answer in the bank".

With reference to the Blum and Leiss model, typical text book tasks on compound interest begin at the point of the mathematical model. Learners then proceed to a mathematical solution and thereafter interpret the solution. They do not operate in the "rest of the world" section of the model apart from producing an interpreted solution and this is trivial. Consequently, they do not see the compound interest formula as a model. In truth, they ***can't*** see it as a model – they don't have access to the real world situation which is to be modelled, and ironically, they may not even realise this.

CONCLUSION

The fact that learners may see the compound interest formula simply as a means to calculate future values is a result of the way in which financial mathematics has come to be constituted in school mathematics. The important issue of daily interest calculation and monthly compounding is not given sufficient attention in the school curriculum, and the tasks offered in school mathematics mask the realities of the real world. Consequently, learners are unable to appreciate the compound interest formula as a model of a tedious daily process in the bank. Furthermore, they will then miss the power of the formula in terms of its efficiency and accuracy.

This work is supported financially by the Thuthuka programme of the National Research Foundation. Any opinions, findings and conclusions or recommendations expressed are those of the author and the NRF does not accept any liability in regard thereto.

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A pilot study exploring pre-service teachers understanding of the relationship between $0,\bar{9}$ and 1

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Studies in the teaching and learning of fractions reveal that learners have considerable difficulty in understanding the relationship between a rational number and its decimal expansion. This qualitative study investigated forty four pre-service students understanding of this relationship at a South African university. Mathematics performance of high school pre-service students was observed after the implementation of activity sheets based on APOS Theory and carried out using the ACE Teaching Cycle. The class of students was divided into groups and worked collaboratively. The findings showed that the APOS designed worksheets had a positive influence on their deduction of the equality $0,\bar{9} = 1$.

INTRODUCTION

Some studies, for example (Weller, 2009; Conradie, 2009, Maharaj et al., 2006, 2007) analysed student mathematical learning on fractions. Our study replicated the study carried out by Weller, et al (2009) in the United States of America. We employed ideas from that study in a South African context. The activity sheets designed for this project were different, as computer aided activities were not used. Before actually carrying out the main study we investigated the validity of the activity sheets in terms of the design. It is this preliminary investigation that is reported on in this paper. In another South African study, Conradie, et al. (2009) asked prospective mathematics teachers to explore whether $0,\bar{9}$ was equal to 1. They found that the majority of students initial response was that "$0,\bar{9}$ was less than 1". However, after discussions some students came up with $0,\bar{9} = 1$. The procedure adopted by that study was to provide four arguments that show that a recurring decimal like $0,\bar{9}$ equals a rational number like 1. Our study on the other hand decided to design certain tasks that would lead students to deciding on the question: Is $0,\bar{9} = 1$? The activity sheets are based on APOS Theory.

POLICY DICTATES IN A SOUTH AFRICAN CONTEXT

The Norms and Standards for Educators (DoE, 1999) expects that the educator be well grounded in the knowledge relevant to the occupational practice. She/he has to have a well-developed understanding of the knowledge *appropriate* to the specialism. Many mathematics educators find themselves in a position requiring them to implement the syllabus, which includes certain topics they are unfamiliar with. According to Adler (2002), educators with a very limited knowledge of mathematics need to develop a base of mathematical knowledge. They need to relearn mathematics so as to develop conceptual understanding. Taking this into account we attempted to make certain that trainee-teachers leave with a base of knowledge relevant to their occupational needs. Mwakapenda (2004) concurs when stating that a significant concern in school mathematics is learning with understanding of mathematical concepts. The National Curriculum Statement (NCS) emphasises a learner-centred, outcomes-based approach to the teaching of mathematics to achieve the critical and developmental outcomes (DoE, 2003). The following question guided our inquiry into pre-service students' understandings of the relationship between $0,\overline{9}$ and 1**:**

> *How does the implementation of an "APOS theory designed activity sheet" and the "ACE Teaching Cycle" facilitate students' learning process with regard to deducing a relationship between* $0,\overline{9}$ *and 1?*

The main intention of the study was to observe how learning of mathematics content, whether effective or not, took place under these circumstances.

THEORETICAL FRAMEWORK

This study is based on APOS Theory (Dubinsky & McDonald, 2001). APOS Theory proposes that an individual has to have appropriate mental structures to make sense of a given mathematical concept. The mental structures refer to the likely actions, processes, objects and schema required to learn the concept. Research based on this theory requires that for a given concept the likely mental structures need to be detected, and then suitable learning activities should be designed to support the construction of these mental structures. The authors have used this framework in many of their studies like Brijlall & Maharaj (2011) and Maharaj & Brijlall (2011). These papers report on two different aspects of the project exploring the learning of specific concepts relating to infinite real sequences.

Asiala, et al (1996) proposed a specific framework for research and curriculum development in undergraduate mathematics education which guided our enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of the following three components: theoretical analysis, instructional treatment, and observations and assessment of student learning. According to Asiala et al. (1996), APOS Theory functions according to the paradigm illustrated in Figure 1.

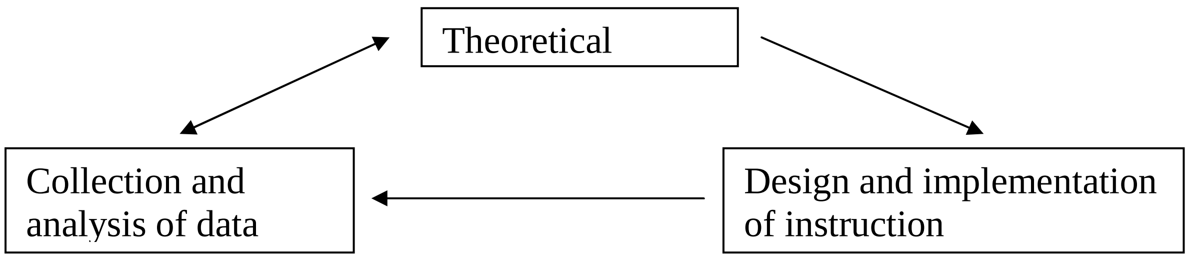


Figure 1: Paradigm: General Research Programme

In this paradigm, theoretical analysis occurs relative to the researchers' knowledge of the concept in question and knowledge of APOS Theory. This theoretical analysis helps to predict the mental structures that are required to learn the concept. For a given mathematical concept, the theoretical analysis informs the design and implementation of instruction. These are used for collection and analysis of data. The theoretical analysis guides the latter, which Figure 1 indicates could lead to a modification of the initial theoretical analysis of the given mathematical concept.

Description of the APOS/ACE Instructional Treatment

APOS Theory and its application to teaching practice are based on two general hypotheses developed to understand the ideas of Jean Piaget. In some studies (see, for example, Weller et al., 2003), these ideas were recast and applied to various topics in post-secondary mathematics. Piaget investigated the thinking of adolescents and adults, including research mathematicians. Those investigations led him to discover common characteristics, specifically certain mental structures and mental mechanisms that guide concept acquisition (Piaget, 1970). According to Dubinsky (2010), APOS theory and its application to teaching practice are based the following assumption on mathematical knowledge and hypothesis on learning mathematics.

- Assumption on mathematical knowledge: An individual's mathematical knowledge is his/her tendency to respond to perceived mathematical problem situations and their solutions by [a] reflecting on them in a social context, and [b] constructing or reconstructing mental structures to use in dealing with the situations.
- Hypothesis on learning: An individual does not learn mathematical concepts directly. He/she applies mental structures to make sense of a concept (Piaget, 1964). Learning is facilitated if the individual possesses mental structures appropriate for a given mathematical concept. If appropriate mental structures are not present, then learning the concept is almost impossible.

The above imply that the goal for teaching should consist of strategies for: [a] helping students build appropriate mental structures, and [b] guiding them to apply these structures to construct their understanding of mathematical concepts. In APOS Theory, the mental structures are actions, processes, objects, and schemas. In the following each of these are briefly described. Then the ACE Teaching Cycle; which constitutes the pedagogical strategies used to follow the hypothesis and the implication for teaching; is described.

After these general considerations, the assumption on mathematical knowlege is focused on by making an APOS analysis of the cognitive relation between an integer or a fraction and its decimal expansion(s). The result of this analysis is called a *genetic decomposition*. A genetic decomposition of a concept is a structured set of mental constructs which might describe how the concept can develop in the mind of an individual (Asiala, et. al., 1996). So, a genetic decomposition postulates the particular actions, processes, and objects that play a role in the construction of a mental schema for dealing with a given mathematical situation.

APOS Theory

The main mental mechanisms for building the mental structures of action, process, object, and schema are called *interiorization* and *encapsulation* (Dubinsky, 2010; Weller et al, 2003). Action, process, object, and schema constitute the acronym APOS. The theory postulates that a mathematical concept develops as one tries to transform existing physical or mental objects. The descriptions of action, process, object and schema; given below; are based on those given by Weller, Arnon & Dubinsky (2009).

Action: A transformation is first conceived as an *action*, when it is a reaction to stimuli which an individual perceives as external. It requires specific instructions, and the need to perform each step of the transformation explicitly. For example, if a student requires an explicit expression to think about a function and can do little more than substitute for the variable in the expression and manipulate it, then such a student is considered to have an action understanding of functions.

Process: As an individual repeats and reflects on an action, it may be *interiorized* into a mental *process*. A process is a mental structure that performs the same operation as the action, but wholly in the mind of the individual. Specifically, the individual can imagine performing the transformation without having to execute each step explicitly. For example, an individual with a process understanding of function will construct a mental process for a given function and think in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs.

Object: If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations (explicitly or in one's imagination), then we say the individual has *encapsulated* the process into a cognitive *object*.

Schema: A mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, called a *schema*. It is coherent in that it provides an individual with a way of deciding, when presented with a particular mathematical situation, whether the schema applies.

Explanations offered by an APOS analysis are limited to descriptions of the thinking of which an individual ***might*** be capable. It is not asserted that such analyses describe what "really" happens in an individual's mind, since this is probably unknowable. Further, the fact that an individual possesses a certain mental structure does not mean that he or she will necessarily apply it in a given situation. This depends on other factors, for example managerial strategies, prompts and emotional states. The main use of an APOS analysis is to point to possible pedagogical strategies. Data is collected to validate the analysis or to indicate that it must be reconsidered. For more details, see Asiala et al. (1996) and Dubinsky and McDonald (2001).

The ACE Teaching Cycle

This pedagogical approach, based on APOS Theory and the hypothesis on learning and teaching, is a repeated cycle consisting of three components: (A) activities, (C) classroom discussion, and (E) exercises done outside of class. Although variations exist, based on the particular topic and local conditions, each iteration of the cycle, in most implementations, takes about one week. The students do all of their work in cooperative groups.

The activities, which form the first step of the cycle, are designed to foster the students' development of the mental structures called for by an APOS analysis. In the classroom the teacher guides the students to reflect on the activities and its relation to the mathematical concepts being studied. Students do this by performing mathematical tasks. They discuss their results and listen to explanations, by fellow students or the teacher, of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems. They reinforce the knowledge obtained in the activities and classroom discussions. Students apply this knowledge to solve standard problems related to the topic being studied.

The implementation of this approach and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies. A summary of early work can be found in Weller et al. (2003).

Collaborative Learning

Our study explored teacher-trainees' understanding, after they carried out investigations first individually and then in a collaborative manner. This is to address the learner-centred approach which underpins Curriculum 2005 (DoE, 2003). We report on an investigation based on the use of worksheets and group-work to construct concepts. To collaborate is to work with another or others. In practice, collaborative learning has come to mean students working in pairs or small groups to achieve shared learning goals (Barkley et al, 2005). Vidakovic (1996, 1997) used APOS theory in the context of collaborative learning. Those investigations focused on the differences between group and individual mental constructions of the inverse function concept.

Genetic Decomposition

A genetic decomposition of a repeating infinite decimal and its relation to a rational number can be fairly simple. At the action level, the student can only list the first few digits of the decimal and may or may not begin to see a repeating cycle. A process understanding of a repeating decimal emerges as the student can imagine writing out all of the digits of the decimal and see that there is a finite sequence of digits that, from some point on, repeats forever to form an infinite string. At the object level, the student sees this string as a totality, and can perform mental or written actions on it.

Design and implementation of instruction

The method adopted four stages: (a) APOS design of activity sheet, (b) Facilitation of control group learning, (c) facilitation of experimental group learning and (d) Interviews. The data collection relied to a large extent on what students could say or write about their learning experiences. The tasks were completed over four double periods, each of one and a half hour duration.. This included the individual work by students, the discussions in the groups, the group class presentations and the final discussion involving the tutor. The interviews will be done with individuals later during the triangulation stage when analysing the data.

Background of students

The students involved were undergraduate teacher trainees from the University of KwaZulu-Natal. They pursue a module on Real Analysis in their final year. This module, which included elementary topology of the real line, involves the learning of concepts in set theory, relations and functions, cardinality, countability, denseness, convergence and other related ideas. The forty five students were divided into a control group comprising twenty four students and the experimental group with twenty one students.

Facilitation of group-work

The students were arranged according to their previous semester grades and were matched accordingly. In this way the two groups comprised of members with similar ability levels. The experimental lot had twenty one students and the control lot had twenty four students. The control group was taught in a traditional manner with the lecturer not involved with the design of activity sheets based on APOS Theory. The experimental group was taught by another lecturer who was involved in the APOS Theory design activity sheets and implemented these in this group. Students in the experimental group were divided into seven groups and engaged with the activities individually for approximately fifteen to twenty minutes before coming together into their respective groups. This was to allow students to make contributions when working in a group setting. Each group, after discussing and reaching a collective decision, presented their mathematical ideas to the class. The student facilitators reported on the collective ideas or thoughts of their groups. The students were given time limits set by the facilitator to encourage them to focus on the task on hand. The groups were similar in that they had members with a spread of ability levels. At the end of the group presentations, an intensive classroom discussion including responses from the lecturer led to students establishing the expected responses to the tasks. This paper reports on learning in the experimental group only. We analyse the written responses of students in this group only. We, in subsequent stages of this research, will triangulate the responses with semi-structured interviews.

Instrumentation

Pre-knowledge

At the beginning of the sessions on this investigation, we found it necessary to recall concepts/techniques that were necessary for the activities leading to a realisation to the question "Is $0,\overline{9} = 1$?". We wanted the students to know the difference between rational and irrational numbers at the outset. Without this, we thought, would be a meaningless exercise. We also insisted that they provide examples to illustrate these numbers. We also consolidated the meanings of terminating decimal number, recurring decimal number, a string of a decimal number, the length of a string and a cycle. Students were also introduced to the notation to be used in class by the researcher and the activity sheets. Conversion of rational to decimal numbers and the reverse conversation were dealt with. The tasks actually encountered by students appear in Appendix One. This work was done in a ninety minute session. The last question (task eleven of appendix one) was originally designed to gain responses so as to carry out a comparison at the end of this investigation.

Design of worksheet

Worksheets were designed in accordance with ideas postulated by Asiala et. al (1996). In this theoretical paradigm we devised four activity sheets (see appendix two), each keeping in mind the mental constructions we felt necessary for successful development so as to conclude that $0,\overline{9} = 1$. Each activity was done in a ninety minute session. We outline the learning outcome and the APOS outcome below.

Activity 1

Outcome of activity: To conceive an infinite string of digits comprising a repeating decimal.

APOS outcome: To help students interiorise the action of listing digits to a mental process.

Question two of activity one allows the students to work with repeating decimal numbers. They are expected to find the first nine digits for different recurring decimal numbers. This is done in a repeatable manner so that students consolidate their understanding at an action level of APOS. However, questions three, four and five expected mental procedures on the recurring numbers. This was to assist the students to interiorise these actions. Question six hoped that the students generalise the concepts learned in this activity by using symbolic representation.

Activity 2

Outcome of activity: To perform operations on strings.

APOS outcome: To help students encapsulate infinite digit strings which they conceived as processes into mental objects to which actions could be applied.

In question one of this activity the students were expected to rewrite expanded notation of recurring decimal numbers in compact form. The principle of reversal was used. Reversal we described as the ability to reverse thought processes of previous interiorised processes (Brijlall & Maharaj, 2010). The second part of question one allows for the four operations on the decimal numbers. In question two, they were required to find a repeating decimal number between $0,\overline{67}$ and $0.6\dot{7}$. This required them to apply actions on mental objects of $0,\overline{67}$ and $0.6\dot{7}$. Also arranging in ascending and descending order meant that they derive a schema for working with recurring decimal numbers.

Activity 3

Outcome of activity: To establish a relation between an infinite digit string and it's corresponding fraction or integer.

APOS outcome: To help students coordinate processes into objects.

In these question we provided opportunity for more operations on recurring decimals. The difference here, however, was to create a platform for "seeing" the operated entity as an object in it's own right. So, a + b the sum of two recurring decimal number must be looked upon as a single decimal number having similar properties. Question three is a crucial one in this activity as it allows for perceiving $0,\overline{9}$ as a sum of two different recurring decimal numbers.

Activity 4

Outcome of activity: To establish a relation between an infinite digit string and it's corresponding fraction or integer and locate it on the number line.

APOS outcome: To help students organize actions, processes and objects into a coherent schema that allows for the unification of $0,\overline{9}$ and 1.

This activity uses a number line model to allow for student identifying the location of decimal numbers. We intentionally work from locating 0,9 on the interval [0;1] and then 0,99 on [0,9;1]. The idea here was to then complete question 3 and hopefully deduce that as we subtracted smaller numbers from 1 the number was approaching $0,\overline{9}$**.** We hoped finally that the conclusion to question 3 in activity four would lead to a realisation of $0,\overline{9} = 1$.

ANALYSIS OF DATA AND WRITTEN RESPONSES

We asked the question Is $0,\overline{9} = 1$ in the pre-knowledge stage and at the end of activity four. We obtained the following results:

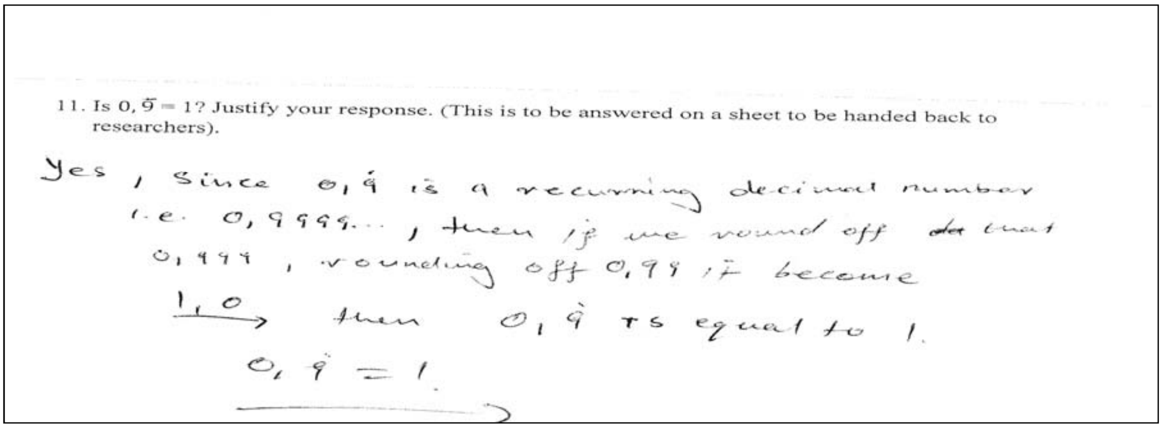
Table 1: Data illustrating post- and pre-activity sheets engagement

Written response	Yes	No	Unsure
Prior to APOS interrogation	1	20	0
After APOS interrogation	11	7	3

Just over fifty percent of the students now indicated that $0,\overline{9} = 1$ as compared to one in twenty one before the implementation of the worksheets. This could mean that the worksheets had a positive overall improvement in mathematical correctness. However, many students still felt that this equality was not true. After the implementation of the worksheets there were students who were now unsure of the result. We shall analyse some episodes of written work for each of these categories.

Those who responded “Yes” to the question

In this category we found that there were students who indicated that $0,\overline{9} = 1$. However, not all students could justify their conclusion. The following is an example (see Figure 2) of such a case:


11. Is $0,\bar{9} = 1$? Justify your response. (This is to be answered on a sheet to be handed back to researchers).

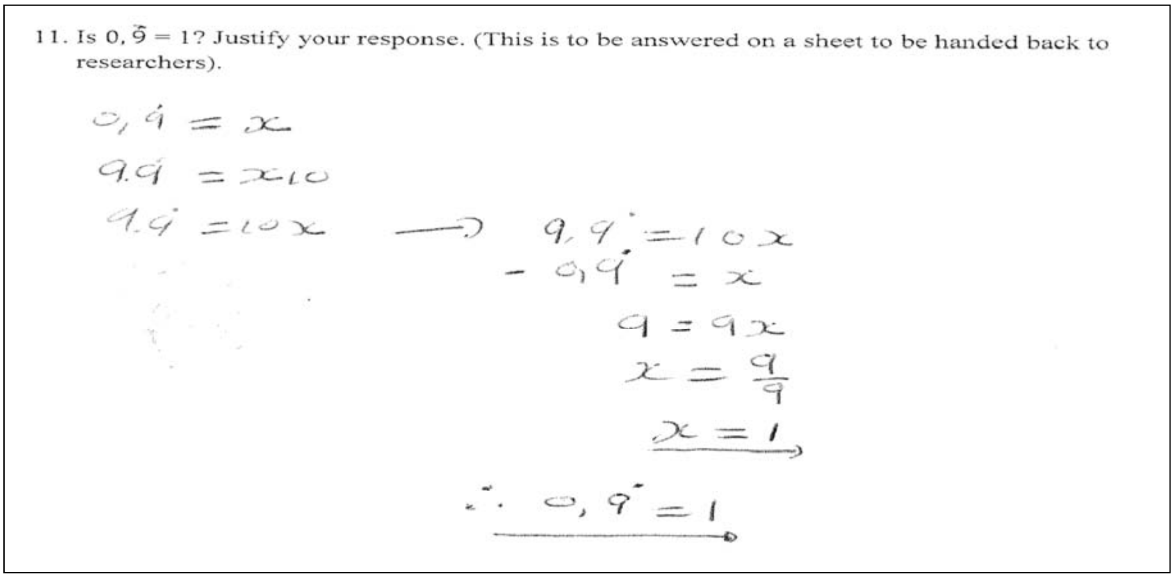
Yes, since $0,\dot{9}$ is a recurring decimal number i.e. 0,9999..., then if we round off that 0,999, rounding off 0,99 it become 1,0, then $0,\dot{9}$ is equal to 1.

$0,\dot{9} = 1$

Figure 2: An example of a student's response with unreasonable justification to why $0,\overline{9} = 1$

This reasoning appeared often and it is clear that these students thought that rounding off (in this case to any number of decimal places) would result in 1. However, it is a problem that these students will now move on to teach in schools with the misconception that a number is equal to the rounded off number when it is actually an approximation to the said number of places. We notice that this type of "approximation" misconception arises in other learning experiences ,as well. One such situation is when students use $\frac{22}{7}$ as an approximation for π. Obviously these two entities are not equal but students inherently accept this equality as instructors say/write $Area\ of\ circle = \pi r^2 = \frac{22}{7}.(2\ cm)^2$ to calculate the area of a circle of radius *2 cm.*

There were students in this category who provided mathematically correct reasons to why $0,\overline{9} = 1$. An example of this follows in Figure 3


11. Is $0,\bar{9} = 1$? Justify your response. (This is to be answered on a sheet to be handed back to researchers).

$0,\dot{9} = x$

$9.\dot{9} = x10$

$9.\dot{9} = 10x \rightarrow 9,\dot{9} = 10x$

$- 0,\dot{9} = x$

$9 = 9x$

$x = \frac{9}{9}$

$x = 1$

$\therefore 0,\dot{9} = 1$

Figure 3: An example of a mathematically sound reason for $0,\overline{9} = 1$

Here the individual reflected on operations (multiplication and subtraction) applied to a particular

process, and he/she is aware of the process as a totality, and constructed such transformations to see $0,\overline{9}$ and 1 as individual entities and being identical. She/he is hence thinking of this process as an object.

Those who responded "no" to the question

The following illustrates a response where the student negated the equality of 1 and $0,\overline{9}$.

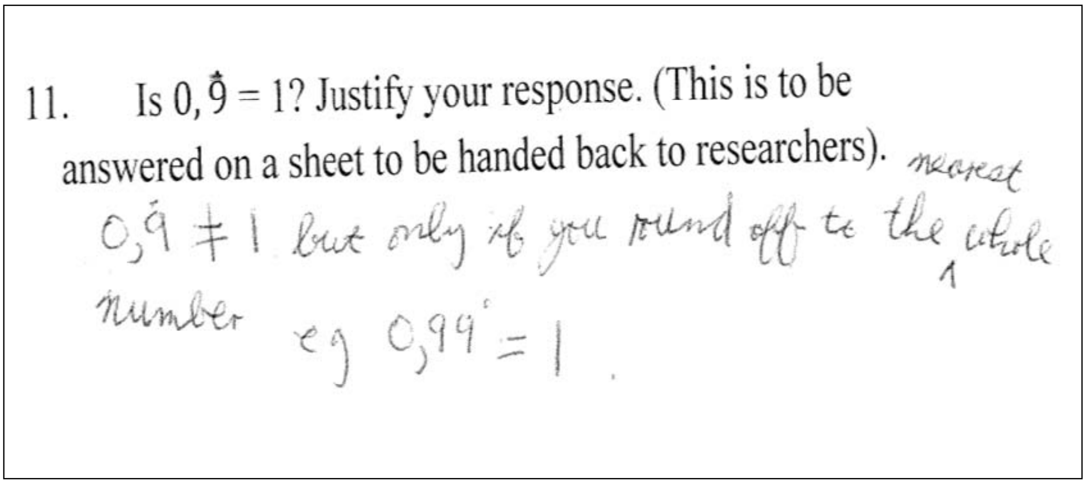
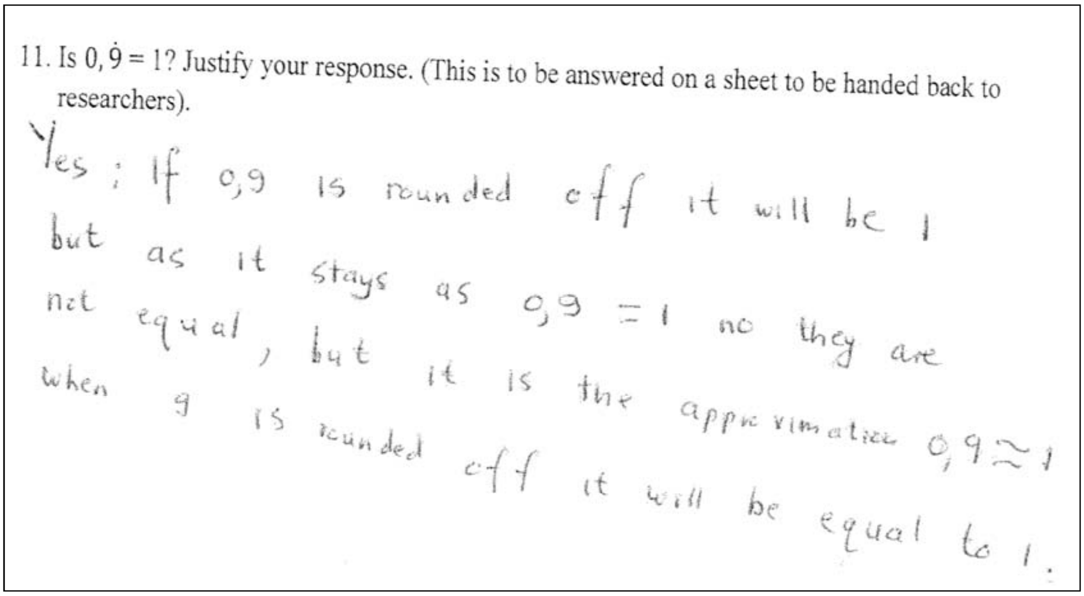


Figure 3: Student who believes $0,\overline{9} = 1$ is false

This student seems to know that rounding off leads to an approximation and on approximating $0,\overline{9}$ she/he gets 1. But, she/he thinks that they are not identical. These mental structures could have developed whilst the student attempted the tasks in the activities. Activity one (see Appendix one) stimulated the student's move from a decimal entity to a whole one. The equality of these entities could have developed during activity four (see Appendix two) which intended the development of the establishment a relation between an infinite digit string and it's corresponding fraction or integer and locate it on the number line. Here the student has applied specific mental structures to make sense of the equality of $0,\overline{9}$ and 1. Another example supporting this thought is indicated in Figure 4 below.



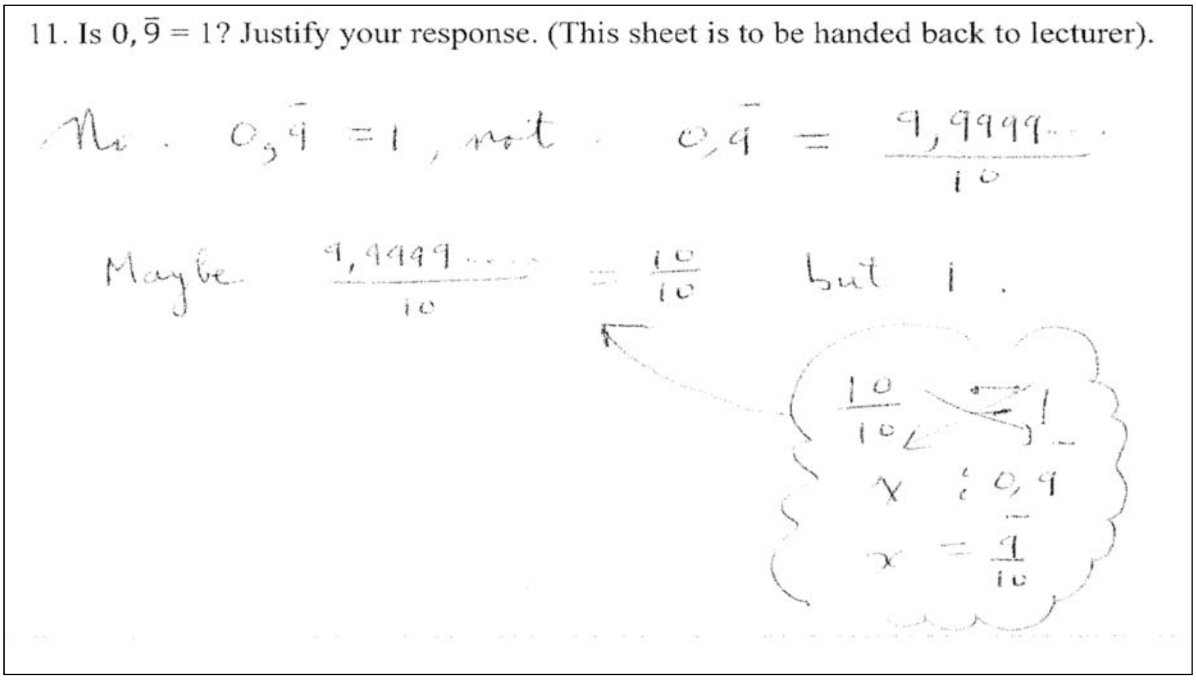
11. Is $0,\dot{9} = 1$? Justify your response. (This is to be answered on a sheet to be handed back to researchers).

Yes; If 0,9 is rounded off it will be 1 but as it stays as 0,9 = 1 no they are not equal, but it is the approximation 0,9 ≃ 1 when 9 is rounded off it will be equal to 1.

Figure 4: An example of equality and approximation

Those who were unsure

Whenever students used the words “maybe” or “sometimes” we classified the response as unsure as a degree of doubt is conveyed. An example of this is illustrated in Figure 5.



11. Is $0,\overline{9} = 1$? Justify your response. (This sheet is to be handed back to lecturer).

No. $0,\bar{9} = 1$, not. $0,\bar{9} = \frac{9,999...}{10}$

Maybe $\frac{9,999...}{10} = \frac{10}{10}$ but 1.

$\frac{10}{10} \simeq 1$

$x = 0,\bar{9}$

$x = \frac{9}{10}$

Figure 5: An example of a response with doubt whether $0,\overline{9} = 1$

This response indicates an error and a misconception. The misconception of approximation and equality arose in the second part of the written response. No clear explanation occurred for the first part of $0,\overline{9} \neq 1$. This would need further investigation. The pilot study hence finds it necessary to carry out interviews in the further study. This would allow for such a response to ascertain reasonable explanation to such a response.

CONCLUSION

The findings of this study showed that some students demonstrated the ability to make use of technique from the worksheets (see Figure 3, for instance). The reference to the equality in the question was essential but ignored by the students. We also found that a greater number of students indicated that $0,\overline{9} = 1$ after the implementation of the worksheets. These activity sheets were designed in accordance with the essential features of APOS theory. It seemed hence that the APOS designed worksheets had a positive influence on their deduction of the equality based on the assumption that an individual does not learn a mathematical concept directly, but applies mental structures to make sense of the concept. However, we need to validate the responses and we are currently involved with a series of interviews to verify our findings.

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THE CONGRUENCE BETWEEN FIRST-YEAR STUDENTS' MATHEMATICAL KNOWLEDGE AND MATRIC PERFORMANCE: AN EXPLORATORY STUDY

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The purpose of this correlational study was to explore how first-year mathematics students' performance in mathematics at matric relates to their performance in a first-year (pre-calculus) mathematics course. The sample comprised 90 first-year students who had registered for maths 1 (a pre-calculus maths course) at the Wits School of Education for the year of this study. Spearman's correlation indicated a statistically significant correlation between participants' maths scores at matric and their maths 1 test scores ($r = 0.36$, $p < 0.01$). The conclusion drawn tentatively was that performance at matric mathematics is a good indicator of performance in first-year pre-calculus maths course. Based on these findings, some recommendations are offered.

INTRODUCTION

It would seem obvious to suggest that the goal of mathematics teacher education is to make teachers-in-training successful both as learners and as teachers of mathematics. But there can be wide diversity of opinions as to what *success in mathematics learning* would mean. For nearly the first half of the twentieth century, for example, success in learning the mathematics of pre-kindergarten through to eighth grade meant facility in using the computational procedures of arithmetic. Also, reform movements of the 1980s and 1990s saw the introduction of such words as 'developing mathematical power' – terms interpreted to mean facility in reasoning, solving mathematical problems, connecting mathematical ideas, and communicating mathematics effectively to others. In the more recent time, successful mathematics learning has acquired composite, comprehensive view and has been described as being mathematically proficient, which, according to Kilpatrick, Swafford and Findell (2001, p. 116), consists of five intertwined components (i.e. conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition).

In the analysis of participants' mathematical knowledge that we describe in this study, no claim that our measurement instrument captures such comprehensive conception of mathematical knowing as proposed by Kilpatrick et al. (2001) or its earliest forms as articulated above is implied. Rather, we admit that our analysis represents nothing more than a description of participants' performance in a first-year mathematics course constitutive of many pre-calculus concepts (e.g. power function, piece-wise function etc) and geometry. Having said that, our knowledge of the structure of the National School Certificate (NSC) examination from which participants' matric results were generated, and the first-year mathematics course that we teach, we believe that performance in these areas could provide some insight into participants' mathematical knowledge in the sense that Kilpatrick et al. (2001) have proposed.

SIGNIFICANCE OF THE SUDY

Reports about first-year university students' difficulties with first-year mathematics courses in many institutions of higher learning are beginning to dominate discussions in the research literature in recent times (e.g. Nampota, 2008; Wolmarans, Smit, Collier-reed & Leather, 2010). Although each university may have its own entry requirements meant to be met by prospective first-year students, reports such as these inevitably invoke questions about how effective these entry requirements have been. In particular, it would be worth knowing how performance in mathematics at matric is related to success with first-year university mathematics, and it is in this area that this study has relevance in contributing to the research literature as well as to the mathematics community. Knowledge such as this is important for making decisions not only about the university entry requirement, but also about the link that should exist between school mathematics and first-year university mathematics.

REVIEW OF RELATED LITERATURE

A disturbing theme gradually gaining prominence in the research literature, in the last few years, is the poor mathematical performance of first-year students in institutions of higher education. Admittedly, this problem is not peculiar to South Africa (see Nampota, 2008, for example). But, for a country like South Africa, where "the enduring shortage of matriculants with good passes in mathematics" (Simkins, 2010, p. 1), is reported to be constraining development in many fields, the situation can be very disconcerting.

In a study that was conducted in the University of Cape Town, Wolmarans et al. (2010) reported that there was a consistent decrease both in the midyear mathematics pass rate and the midyear mathematics mean mark for the South African first-year students involved in their study. In an earlier study carried out in Malawi, Nampota (2008) bemoaned the situation where many of the first-year students in her study indicated that their high school mathematical experience ill-equipped them for a successful study of university mathematics.

Clearly, the first-year mathematics students in the university are certainly those who, through their mathematical performance at matric, have been certified qualified with potentials for a successful study of university mathematics. It worries, then that many of these supposedly qualified first-year students encounter severe difficulty with university mathematics, and this calls for an empirical investigation. Our goal, though, is not to interrogate the predictive efficacy of matric results in connection with the successful study of university mathematics by prospective first-year students. Instead, we aim to explore the question of whether performance in mathematics at matric is related to performance in first-year mathematics course.

METHOD

Consistent with its declared goal, this study employs a correlational research design, which according to Schunk (2004), usually generates quantitative data to explore relations that may exist between variables. Durrheim (1999) further reiterates the point that in correlational studies, such as ours, correlation coefficient is a more exact way of representing relationships between constructs.

The sample

The sample for this study comprised 90 registered first-year mathematics students for the study year at the School of education, University of the Witwatersrand, South Africa. We employed convenient sampling procedure to constitute the sample for this study, since the participants form a part of the class of students to whom we teach mathematics 1 (hereafter, maths 1) – a first-year mathematics course consisting of algebra and geometry.

Data collection procedures

Participants provided two sets of data for this study. The one set consisted of participants' test scores obtained in the maths 1 course that we teach, and the other set was their individual mathematics scores at matric. During the testing time, participants were requested to provide their marks in mathematics at matric. Some provided these marks in letter symbols (as received from their respective schools, e.g. A, B, level 5 etc), and there were a few others who did not supply their matric marks, since participation was voluntary. These two groups of students (about 20 of them) were thus excluded from the study. Therefore, the 90 students involved in this study were those who supplied both sets of data – marks at matric and test scores. This number, (90 students), in our view, is a fair representative sample, since according to Borg and Gall (quoted in Cohen, Manion & Morrison, 2007, p. 102), "correlational research requires a sample size of no fewer than thirty cases".

Spearman's correlation was calculated using the two sets of scores in order to establish a relationship. The results are presented in the next section, but first, we acknowledge some of the limitations inherent in our procedures that call for caution when interpreting our results.

Although the majority of our sample consisted of South Africans, we did not account for the impact that the performance of the few international students who might have taken part in this study may have had on the results. The study could not also guarantee that the results would have remained the same had the few students who were excluded from the study on the basis of supplying their marks in symbols participated.

THE RESULTS

For the ease of reference, the results are presented in two separate tables. Table 1 represents a descriptive summary of the findings, while Table 2 reflects Spearman's correlations.

Table 1. Mean percentage scores of participants in matric maths and in maths 1 test

	N	Mean score	Std. dev.	Max. score	Min. score
Matric score	90	67.1	12.2	92	37
Test score	90	52.1	16.8	96	13

Table 2. Correlation between participants' matric maths scores and their maths 1 test scores

	N	Mean score	*r*-value	*p*-value
Matric score	90	67.1	0.36	0.000
Test score	90	52.1		

DISCUSSION

The purpose of Table 1 is not to compare participants' mean score at matric with their mean score in the maths 1 test, since the tests from which the scores were generated are not necessarily similar in terms of breadth and depth. Instead, the mean scores must be interpreted in relation to the respective tests.

Looking at Table 1, one is easily led to conclude that the participants, given their mean marks, have a fair, acceptable knowledge of mathematics. While we do not dispute that this is probably the case, caution still needs to be exercised when interpreting these results. The very large standard deviations shown in Table 1 indicate that the scores are not homogeneous (hence, our use of Spearman's correlation), and hence, the mean scores may reflect the mathematical ability of only a few of the participants in this study (see Daramola, 1998). A consideration of the range (maximum score - minimum score) for the respective sets of scores in Table 1 buttresses this point. Although we could not establish that the student with the minimum mark of 37% in matric was the same student who had 13% (the least mark) in the maths 1 test, it is worth asking whether 37% in maths at matric is reasonably fair mark for a student expected to be successful with the study of university mathematics.

The Spearman's correlation represented in Table 2 indicates that there is a moderate positive correlation between participants' scores in matric maths and their maths 1 test scores. Table 2 particularly indicates that the correlation between performance in maths at matric and performance in maths 1 is statistically significant ($r = 0.36$, $p < 0.01$). What this translates to is that a student with a good pass at matric maths is likely to obtain equally good pass in maths 1 test or examination. Furthermore, the association of the very high significant level ($p = 0.000$) with a moderate correlation coefficient ($r = 0.36$) could be interpreted to mean that the chance that performance in maths at matric is moderately related to performance in maths 1 is very high.

CONCLUSION AND RECOMMENDATION

The results as presented and discussed above have important implications for the successful study of university mathematics by first-year students. If performance at matric maths implies performance in university first-year maths, as the findings from this study tend to support, then university entry requirement may need to be given greater attention than it currently receives if prospective first-year mathematics students are expected to be successful in their study. Second, first-year mathematics student trainers would equally need to be privileged with information regarding the matric maths information of their students. This could help them to make decisions about where to begin teaching and remediation.

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CASE STUDIES OF TEACHER DEVELOPMENT ON A MATHEMATICAL LITERACY ACE COURSE

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In this study I focused on Mathematical Literacy teacher development within the field of in-service training practice. I investigated two teachers' understandings of Mathematical Literacy and Mathematical Literacy teaching practices within an ACE Mathematical Literacy course, focusing on the relationship between context and mathematical content. The data collected included ACE course tasks and videotaped observations of Mathematical Literacy teaching over time. The study showed limited, but visible shifts in understanding with respect to meaning and practice. Key findings relate to assessment, an adaptation to the spectrum of Mathematical Literacy teaching agendas (Graven and Venkat, 2007) and the emergence of a moral value-based dimension in Mathematical Literacy teaching practice.

BACKGROUND TO THE STUDY

Mathematical literacy is a new subject area which developed largely because of concerns with respect to the low numerical literacy levels of adults. In 2006, Mathematical Literacy[32] was introduced as an optional school subject in South Africa in the FET band (Grades 10-12). In South Africa, in particular, the new subject still needs to find its place with respect to nature, purpose and teaching practice.

Mathematical Literacy teachers appear to be unsure about the interpretation of the Mathematical Literacy curriculum and how it should be taught and assessed (Graven and Venkat, 2007; Christiansen, 2006, 2009; Venkat et al., 2009; Vithal and Bishop, 2006). In bridging the gap between curriculum demands and Mathematical Literacy teachers' knowledge, understanding and practice, a re-skilling course for practicing teachers was developed at an urban university. The ACE Mathematical Literacy course was designed as a pilot training programme focused on the 're-skilling' of in-service teachers from a range of subject areas to develop as Mathematical Literacy teachers. The pilot programme for the ACE Mathematical Literacy course ran from 2007-2008 and involved nineteen teachers from areas that might be described as 'previously disadvantaged' areas. Seven of the teachers taught Mathematical Literacy in their schools in 2007.

The ACE Mathematical Literacy course aimed at enabling teachers to critically analyse and interpret the Mathematical Literacy curriculum in order for them to develop an understanding of the curriculum and to discuss and develop a sound

[32] Mathematical Literacy refers to the school subject in South Africa; mathematical literacy refers to the general notion of numeracy, quantitative literacy, mathemacy or functional mathematics.

practice for the teaching of Mathematical Literacy. The course was organized across four modules and mathematical content was presented within real-life contexts.

One of the modules, Introduction to Mathematical Literacy, aimed to build teachers' understandings of the connection between the components of mathematical content and real-life context, and to emphasize this link within Mathematical Literacy teaching practice. The module focus was on the development of skills for planning, teaching and assessment, using a range of teaching and assessment strategies.

The aim of my research study was to evaluate the ACE teachers' development of meaning and understandings with respect to Mathematical Literacy curriculum interpretation and Mathematical Literacy teaching practice over the duration of this module. This paper focuses on the following research questions that were investigated as part of the study:

Question 1: How does the ACE Mathematical Literacy teacher[33] teach Mathematical Literacy in the classroom? In particular: How does the teacher work with the relationship between mathematical content and context, and how does this change over time?

Question 2: What can be said about the types of questions and the cognitive levels of questions teachers develop for assessment tasks, and how does this change over time?

THEORETICAL FRAMEWORK

Wenger's (1998) social theory of learning was utilized as the theoretical framework for this research study. The learning of an individual includes both individually constructed intellectual learning and learning as participation embedded within a community. This community for learning is defined by Wenger (1998) as a community of practice.

Communities of practice

The empirical data for this study was collected within the community of learning of the ACE Mathematical Literacy course participants. Two main communities of practice were involved in this study, namely the interactive ACE Mathematical Literacy classroom and the ACE Mathematical Literacy teacher's school classroom. The ACE Mathematical Literacy course aimed at alignment between the two communities of practice. Given that almost all the ACE Mathematical Literacy course teachers began the course with no prior experience of teaching Mathematical Literacy, the community of practice constituted by the ACE Mathematical Literacy classroom formed the key community of practice in relation to the teachers' learning

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[33] The term 'teacher' in ACE Mathematical Literacy teacher refers to an in-service teacher who was enrolled as a student on the ACE Mathematical Literacy course

and development.

Wenger's four components of learning

Wenger's learning theory (1998) distinguishes four components of learning, namely: meaning, practice, community and identity. He notes that these elements of learning are 'deeply interconnected and mutually defining' (Wenger, 1998, p5). The course aimed at supporting the development of the teachers' understandings of the Mathematical Literacy curriculum and to enable them to develop Mathematical Literacy teaching practices aligned with these developing understandings. In the research study I focused on Wenger's learning components of meaning and practice.

Meaning

The meaning component was studied in terms of the ACE teachers' understandings and experience of the new Mathematical Literacy curriculum as meaningful, as well as their talk about ways of teaching the curriculum. Emerging within the meaning category was reference to ownership of what it meant to be mathematically literate, in other words to understand and voice the nature and purpose of the subject in the teachers' own words. The study focused on tracing the ACE teachers' changing understandings over time as a key indication of learning as the ACE course and communities of practice developed.

Practice

Data on the component of practice was collected through focusing on lesson planning, the development of assessment tasks and snapshots of Mathematical Literacy lesson presentation involving contextual data on smokers' statistics.

According to Wenger's theory (1998), the understanding and experience of the Mathematical Literacy curriculum and its outcomes as meaningful by the teacher will lead to the establishment of classroom practice, or to the change of an existing practice, and vice versa. The empirical data for the study indicated that the ACE Mathematical Literacy teachers' meanings and practices were not straightforwardly linked.

LITERATURE REVIEW

The nature and purpose of mathematical literacy

In the literature mathematical literacy is largely described in two ways relating to the nature of the link between content and context. The frame for description could be described as either a contextual or a mathematical frame.

Contextual frames for mathematical literacy were developed by Steen (2001) and by Skovsmose (1992). Steen (2001) claims that citizens can solve authentic, real-life problems with the aid of elementary, simple mathematics, following uncertain procedures. The ability to solve these real-life problems would enable citizens to make competent, responsible decisions in order to function in the world. Skovsmose (1992), on the other hand, follows a more reflective, critical line when he suggests the use of mathematics and technology for solving problems set in real-life contexts. His aim is the development of critical citizens who can reflect on the mathematical solutions of real-world problems in order to develop equal societies where justice prevails. Pugalee's (1999) mathematical frame for mathematical literacy sharply contrasts with the views of Steen (2001) and Skovsmose (1992) since his emphasis falls largely on the mathematics.

In South Africa teachers may choose to work in either of the two frames. When a teacher works in a contextual frame the emphasis falls on the context and the mathematics is backgrounded, and vice versa.

The South African Mathematical Literacy curriculum documents (DoE, 2003; DoE, 2005; DoE, 2008; DoE, 2006) appear to be a hybrid of the contextual and mathematical orientations. Three main Mathematical Literacy teaching approaches are visible within the curriculum documents:

1) A teaching practice where the focus is on the context and the mathematics is used to solve a contextual problem (DoE, 2003)
2) A teaching practice where the context and mathematical content appear to be presented in an interconnected manner (DoE, 2005; DoE, 2008; DoE, 2006)
3) A teaching practice where the focus is on the mathematical content and the context is used as a vehicle to teach the mathematics (DoE, 2003)

It might be noted that the curriculum documents convey mixed messages (Venkatakrishnan and Graven, 2006, p20) on how to teach Mathematical Literacy, in particular with respect to the relationship between context and mathematical content.

The teaching of Mathematical Literacy

The relationship between context and mathematical content

The 'newness' of the subject with reference to pedagogy and assessment has produced a wide spectrum of curriculum interpretations with respect to the goals and teaching practice of Mathematical Literacy when referring to the link between context and content. The result is that teachers are likely to structure their teaching practice in a direction according to their own interpretation of the policy documents. According to Graven and Venkat (2007) there is empirical evidence of a spectrum of Mathematical Literacy teaching practices, ranging from mathematical to contextual orientations. The context or the mathematical content might be foregrounded alternatively in or across Mathematical Literacy lessons.

The types of questions and the levels of questions

The Subject Assessment Guidelines emphasize the view that when teachers assess Mathematical Literacy the components of content and context should be linked since 'assessment in Mathematical Literacy needs to reflect this interplay between content and context' (DoE, 2008, p7). The document also suggests that 'reasoning and reflection questions' ask for higher-order cognitive skills (DoE, 2008, p 27-28).

Social and moral values

The focus on social and ethical values emerged from the empirical data collected for this study. Some teachers introduced contextual class discussions on the practice of smoking which, in several cases, led to conversations about the reasons for and effects of smoking. The emergence of moral discussions with reference to the practice of smoking was investigated for possible inclusion of a moral, ethical component in the spectrum of Mathematical Literacy teaching agendas (Graven and Venkat, 2007).

Tuana (2007) claims that moral literacy, in addition to language and numerical literacy, are important elements of education. According to Tuana (2007, p2) moral literacy involves three basic components that interact and mutually reinforce one another: ethics sensitivity, ethical reasoning skills and moral imagination.

Ethics sensitivity (Tuana, 2007, p2) includes the awareness of moral intensity and identification of moral virtues of an ethical situation within a community. Ethical reasoning skills (Tuana, 2007, p3) involve critical reasoning and the ability to identify and assess the validity of facts relevant to an ethical situation. Moral imagination refers to the 'image' of probable actions that might develop as a result of the blend of 'reason and emotions' within a given context (Tuana, 2007, p5). This might lead to a sense of 'personal ownership' (Tuana, 2007, p6) of actions and therefore a sense of responsibility. Ownership might suggest possibilities for personal decision-making and actions, which align with the view of the Department of Education with respect to the development of a 'responsible citizen' (DoE, 2003, p2) and a 'self-managing person' (DoE, 2003, p9).

METHODOLOGY

In order to investigate the research questions, I conducted a longitudinal case study (Denscombe, 2007) on two ACE Mathematical Literacy teachers. The trajectory data collected over 16 months as part of the study provided a picture of the ACE Mathematical Literacy teachers' developing understandings and interpretations of the Mathematical Literacy curriculum and evidence of their sense of Mathematical Literacy teaching. The data provided a reflection on the learning components of meaning and practice (Wenger, 1998) over time.

The nine data collection instruments used for this study included formal written ACE assessment tasks, informal written tasks collected for research purposes, interviews and videos of classroom practice. The central data collection tool that was used for the research was based on contextual data on the rate of smoking amongst adults in different countries. All ACE teachers were given statistics on these smoking patterns during the first session of the course. Early in the module they were asked to informally plan a lesson including questions they would ask using the data (Portfolio 1, 20 February 2007). Later, the ACE teachers were requested to use the same data and design a lesson and a worksheet for use in Mathematical Literacy teaching (Assignment 2, 11 September 2007). The video-taped lessons (September, October 2007) of the ACE teachers who taught Mathematical Literacy in 2007 provided a snapshot of their teaching practice using the specific data on smoking.The data for Teacher A and Teacher B were selected at a late stage of data collection and then analyzed in more depth.

KEY FINDINGS OF THE STUDY

Some of the key findings of this study included the development of a range of possible questions for Mathematical Literacy assessment tasks, reference to the cognitive level of questions, an adaptation to the spectrum of Mathematical Literacy teaching agendas as developed by Graven and Venkat (2007) and the emergence of a moral value-based dimension in Mathematical Literacy teaching practice.

The classification of Mathematical Literacy questions

The empirical data collected for the study indicated that the teachers included contextual and mathematical questions in classroom assessment tasks, but that these questions were largely not structured to enhance the connection between context and mathematical content. The use of the spectrum of Mathematical Literacy teaching agendas (Graven and Venkat, 2007) led to an analytical frame that could be used for thinking about types of Mathematical Literacy questions. The frame presents a classification that includes five categories of questions that relate to Mathematical Literacy teachers' interpretation of the relationship between context and content (Hechter, 2011, p145). The classification of questions might be included in the spectrum of Mathematical Literacy teaching agendas developed by Graven and Venkat (2007).

The categories and description of the categories are the following:

Contextual questions: These questions are purely contextual questions with the focus on the investigation of the context. No reference is made to the mathematics; the mathematics is not in service of context. These questions are not included in the spectrum of teaching agendas (Graven and Venkat, 2007).

Contextual questions where the mathematics is in service of the context: These questions are contextual questions, but the mathematics embedded in the context is used to inform and solve the contextual situation. These questions could mainly be found in the 'Context driven' and 'Content and context driven' agendas (Graven and Venkat, 2007).

Dialectical questions: These questions are asked to deepen both the mathematical understanding and the contextual understanding; the mathematics informs the context, and vice versa. These questions could mainly be found in the 'Context driven' and 'Content and context driven' agendas (Graven and Venkat, 2007).

Mathematical questions where the context is in service of the mathematics: These are mathematical questions, but the questions use a contextual frame within which data is located and used to do the mathematics. The context is used as a vehicle within which mathematics is done. These questions are similar to the traditional word problems. These questions could mainly be found in the 'Mainly content driven' agenda (Graven and Venkat, 2007).

Mathematical questions: These questions are purely mathematical with no reference to the context. The focus is on doing the mathematics; the context is not in service of the mathematics. These questions could mainly be found in the 'Content driven' agenda (Graven and Venkat, 2007).

The empirical data indicated that the types of questions included by Teacher A and Teacher B could either be classified as contextual questions or mathematical questions where the context is in service of the mathematics.

The cognitive level of Mathematical Literacy questions

The empirical data showed that the cognitive levels of questions were by and large set at a relatively low level. Furthermore, the data indicated the inclusion of 'low-level reflective questions', but no 'reasoning and reflecting questions' (DoE, 2008). The analysis of the cognitive level of questions suggested the addition of 'low-level reflective questions' (Hechter, 2011, p145) to the existing Mathematical Literacy assessment taxonomy (DoE, 2008, p8). 'Low-level reflective questions' are questions that require contextual reflection based on personal opinion or experience; no mathematical reasoning or calculations are required to answer these questions.

Adaptation of the spectrum of Mathematical Literacy teaching agendas (Graven and Venkat, 2007)

Teacher A worked partially within the 'Context driven' agenda and the 'Mainly content driven' agenda (Graven and Venkat, 2007), and Teacher B's teaching practice could be placed in an adapted version of the 'Mainly content driven' agenda (Graven and Venkat, 2007).

Teacher A's teaching practice suggested the need for the possible inclusion of an additional agenda which focuses on the engagement and investigation of the context without engagement with the embedded mathematics (Hechter, 2011, p146). The 'Context driven (without mathematical connections)' agenda could be placed to the left of the 'Context driven' agenda (Graven and Venkat, 2007). This agenda will include Contextual questions (Hechter, 2011, p145). It could be argued that this agenda is not suitable for Mathematical Literacy teaching practice and should not be included in the spectrum of teaching agendas since it focuses on the discussion of context without engaging with the mathematics. However, this option may not be ignored since it followed from Teacher A's empirical data.

It might be suggested that the 'Mainly content driven agenda' (Graven and Venkat, 2007) is adapted to include the teaching practice followed partially by Teacher A and by Teacher B. The current agenda aims to 'learn maths and then to apply it to various contexts'. It was seen in the empirical data that both the ACE teachers engaged with the context (in different ways and to different extents), and then followed the discussion of the context with the doing of mathematics using the contexts as a vehicle to do the mathematics, therefore to 'learn the context and then to use the context to do the maths'. Answers to the calculations would not necessarily be linked back to the context again. The context and the mathematical contents were largely dealt with separately and a dialectical relationship did not develop between the components. It is suggested that the 'Mainly content driven' agenda should be adapted to include both directions as explained above (Hechter, 2011, p146).

Ethical and moral values

The empirical data indicated the presence of ethical and moral value discussions in the Mathematical Literacy classroom. These discussions could support the development of the elements of moral literacy (Tuana, 2007). It is feasible that class discussions, if the context calls for it, lead to more than the development of ethics sensitivity and awareness with respect to a moral issue, but is extended so that the teacher and learners identify, assess and reason about facts and values in order to develop ethical reasoning skills and possibly their moral imagination.

The spectrum of Mathematical Literacy teaching agendas (Graven and Venkat, 2007) may also be expanded to include reference to ethical and moral values discussions (Hechter, 2011, p147). These value discussions will, if the context calls for it, feature more prominently in the newly described 'Context driven (without mathematical connections)' and the 'Context driven' agendas, but may also be included in the 'Content and context driven' agenda.

CONCLUSION

This study suggests the need for Mathematical Literacy teacher training so that teachers can make a coherent connection between contexts and the mathematical content required for solving problems rooted in real-life contexts (Hechter, 2011, p149). More attention needs to be given to the notion of 'situational sense-making and contextual orientation' (Venkat, 2010, p57), therefore understanding the real-life situation and the embedded mathematics.

Mathematical Literacy teacher training needs to include aspects of curriculum implementation, lesson planning, presentation and assessment.

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EXPLORING THE PROMOTION OF MATHEMATICAL PROFICIENCY IN A GRADE FIVE CLASS

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A case study was conducted in one school in an informal settlement in Gauteng. The study explored how a grade 5 teacher promoted mathematical proficiency in a mathematics class of English second language learners. The study looked at the kind of tasks the teacher used in her teaching and whether they enabled mathematical proficiency. Data was collected through video recording of lessons and the teacher interviews. The study showed that this teacher did not provide enough opportunities for the learners to engage in mathematical thinking`

WHY MATHEMATICAL PROFICIENCY?

The Revised National Curriculum for Mathematics General Education and Training indicates that the mathematics programme should provide opportunities for learners to develop and employ their reasoning skills and be able to evaluate the arguments of others (DoE, 2002). The teacher needs to ensure that the learners are exposed to mathematical practices that promote mathematical reasoning and hence proficiency. Furthermore, Du Plessis et al. (2007) argue that the teacher has to go beyond testing only knowledge but has to encourage learners to make judgments, express their own opinions, and substantiate what they say. She further argues that learners need to be encouraged to notice connections, evaluate problems, search for solutions to problems, create new things and make predictions (du Plessis, 2007).

Mathematical proficiency is not a content area like, say addition, it is a competence that is embedded in the practice of mathematics. It has to be attended to in all the mathematics learning outcomes and a full range of concepts that are taught in each grade and is therefore not easy to achieve. Kilpatrick, Swafford and Findell (2001) argue that mathematical proficiency is the evidence of mathematical reasoning. Mathematical proficiency is evident when learners show conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and a productive disposition (Kilpatrick et al., 2001). I would like to see if the teacher is able to create a mathematical community in which different student responses are listened to and respected. As Ball and Bass (2003, p. 20) argue, “If students are to engage collectively in substantial mathematical work, she needs to learn to listen attentively and critically to the views of others.” Listening attentively helps learners critically evaluate mathematical claims, both their own and that of others and this is a key component in the development of mathematical proficiency.

THE ROLE OF MATHEMATICAL TASKS WHEN PROMOTING MATHEMATICAL PROFICIENCY

It is very important to look at the kinds of tasks given to learners in this research. The kinds of mathematical tasks that the teacher availed to the learners is important when promoting mathematical proficiency. Stein et al. (1996) classify the different kinds of tasks into two, namely, lower-level demand tasks and higher-level demand tasks.

Lower-level demands

Stein et al. (1996) explain lower level demand tasks as memorization tasks and procedures without connections with limited cognitive demand. Memorization tasks require learners to reproduce previously learned facts, rules, formulae or definitions. Procedures without connections are algorithmic, focusing on producing correct answers rather than developing mathematical understanding.

Higher-level demands

These kinds of tasks refer to procedures with connections tasks and doing mathematics tasks. In procedures with connections tasks learners use procedures for the purposes of developing deeper levels of understanding of mathematical concepts and ideas. These kinds of tasks require some degree of cognitive effort. Stein et al. (1996) argue that these tasks require the learners to explore and understand the nature of mathematical concepts, processes or relationships.

Implementation of tasks

The research focused more on the implementation rather than the setup of the task. It explored whether the teacher helped the learners to justify or explain their solution processes when dealing with tasks that are rich enough to afford such opportunities (Stein, Smith, Henningsen and Silver, 2000). Stein et al. (1996) emphasize that while the learners engage in mathematical tasks that promote reasoning the teacher should provide them with sufficient time and provide encouragement for exploration of mathematical ideas.

THEORETICAL FRAMEWORK

The project is broadly informed by the social constructivists' theory of learning (Von Glasersfeld, 1987; Vygotsky, 1978)

While the teacher helps the learners justify and explain their answers, an issue of social interaction arises. This is supported by Von Glasersfeld (1987) and Vygotsky (1978) who emphasize the importance of social interaction in the classroom where the teacher encourages learners to interact with each other to explain their thinking and to justify their answers. From an interactionist's perspective it is important to create a teaching environment in which learners are encouraged to talk about their mathematical understandings with each other thereby developing their mathematical thinking (Jaworski,1994), and hence mathematical proficiency.

Von Glasersfeld (1987) argues that the teacher uses language to guide the students' construction. He argues that "the student, working on some mathematical task, talks with the teacher and stimulated by the teacher's prompts and responses with the teacher reveals aspects of awareness which provide clues." (cited in Jaworski, 1999, p. 27). Vygotsky (1978) maintains that with appropriate instruction there may be a potential that the child reaches higher conceptual levels than he/she would be able to achieve on their own.

METHODOLOGY

The research focused on the question: What mathematical tasks and learner interactions does the teacher foster in her classroom in order to develop learners' mathematical proficiency? This study is an explorative case study conducted in one grade 5 mathematics class in an urban setting in Gauteng. Data was collected through video recording. The video focused on the interaction between the teacher and the learners and how the teacher communicated with the learners to promote mathematical proficiency. The video also focused on the kinds of tasks given to learners.

The teacher has a three year diploma and an Advanced Certificate in Education in Mathematics. She has been teaching mathematics for the past six years. Her home language is Venda. She has a reputation of being committed to self-development This grade 5 teacher was teaching about 2 dimensional shapes and 3 dimensional objects. This was not her first lesson about the 2D shapes and 3D objects. She gave each learner a worksheet and read the questions together with the learners; explaining each question verbally in Setswana. I have chosen to focus my discussion of the classroom narrative on the discussion that ensued from the questions presented in figure 1.

a) **What is the name of object A?**
b) **What is the name of object B?**
c) **Can the two 3D objects be used for the same purpose in real life? Give reasons for your answer?**
d) **Write the similarities between the two 3D objects.**
e) **Name the 2 D shapes that make object A?**
f) **Name the D shapes that make object B?**

Figure 1

ANALYSIS OF CLASSROOM NARRATIVES

At the beginning of the lesson the teacher made her behavioral expectations clear to all the learners. She endeavored to create a mathematical community (Wood, Cobb and Yackel, 1992) in her classroom environment by expecting learners to respect and value each other's mathematical responses during class discussions.

After explaining the questions to the learners the teacher did not just tell the learners the names of the given 3D objects. Through questioning the teacher expected the learners to reproduce previously learned definitions. The question required limited cognitive demand as it focused on the learners giving a correct answer from recall, rather than developing mathematical understanding.

6. T: What is the name of this 3D object? (*pointing the pyramid*)
7. L: mh…… he…. I think it's a, it's a prism ma'am
8. T: why do you say it's a prism?
9. L: mh… because e.. it is like a sharp box
10. T: a box? Do all boxes look like this?
11. Class: no

In utterance 8 and 10 the teacher did not just accept the answer 'prism'; she wanted the learners to justify their answer. Wood et al. (1992) argue that learners should be provided with an opportunity to provide information to help clarify the meaning of their answers. Wood et al. (1992) further insist that the teacher is not supposed to impose his/her ways of thinking on learners. When the answer was incorrect in utterance 7 and 9 the teacher asked the learners to clarify and justify their answers. The teacher was challenging the learners' previous knowledge of procedures and facts which were previously learned definitions.

The teacher's demand for the reason in utterance 8 helped her realize that the learner did not really understand what a prism was. This resonates with Schiffer's (2001) recommendation that teachers assess the validity of a learner's mathematical argument rather than just accept the learner's unjustified answer. This helps identify whether the learner just successfully memorized facts and procedures or whether they understand the mathematics. As much as the teacher was constantly asking the learners to explain their answers, her probing was focused on the learners' explanation of previously learned pieces of information. Consequently, the learners were mostly reproducing previously learned facts.

12. T: what is a prism hante (in actual fact)?
13. Class: quiet (*everyone seem to be thinking*)
14. *(The teacher decides to put a model of a prism and pyramid on the table)*
15. T: ok class, look carefully at these two objects, eh, what do you see?
16. L: the first one is flat on top and the other one is sharp on top

17. T: ok, you see, this one is called prism, have you forgotten, this one is pyramid. I am sure you can see that the prism has quadrilaterals only and the pyramid has triangles and a quadrilateral.

In utterance 14 and 15 the teacher was trying to scaffold her questions by using physical models. This suggests that she was trying to break down the concept and introduced a simpler version of the problem. Though in utterances 8, 10 and 15 the teacher was trying to develop productive ways of challenging learners to justify their answers, in utterance 17 she somehow got impatient and decided to give the answer to learners without giving them a chance to struggle first. She did not give the learners a chance to explain the difference between the two objects on their own. She seemed to fail to present the challenging aspects of this task to the learners so that the learners may find the correct answers on their own (Stein et al., 1996).

In utterance 27, the teacher again provided the learners information rather than use questions that would evoke conceptual understanding in the learners. In complex teaching environments, teachers are expected to make decisions about when to provide information, when to clarify an issue, when to model, when to lead, and when to let the learners struggle with a difficulty. Ball & Bass (2003) argue that learners have to be provided with opportunities, encouragement and assistance to engage in thinking, reasoning and sense making on their own in the mathematics classroom.

27. T: okay let's see, now can you see that this is a pyramid? (Showing the learners a pyramid model) Right, what shapes do you see here? ***(Holding a pyramid model)*** Yes Tutu.

28. L: I see a triangle ma'am

29. T: and what else?

30. L: and mh square

31. L: hayi ***(no***) e rectangle.

32. T: why do you say it's wrong? What is this shape? (***pointing at a quadrilateral in question***)

33. L: because e e … a square is not like this.

In utterances 28 to 31 the teacher allowed the learners to express ideas with peers. She managed to listen to the second learner who was opposing the first learner. She however chose not to give learners more time and space to discuss ideas to a conclusion. The learner in utterance 33 could have been asked to elaborate on the differences between a square and rectangle in order to help him reach a conclusion rather than giving him the answer immediately in utterance 34. Alternatively, her role could have been to monitor learners' participation in the discussions and decide when and how to encourage each learner to participate.

34. T: yes you are right, can you see that these sides are not equal and therefore this is a rectangle and not a square.

35. T: Let's go to the next question

The teacher often cut the discussions short rather than let the learners struggle with difficult problems. She would quickly assist learners who were encountering difficulties without giving them a chance to explain. In utterance 34 and 35 the teacher wanted to help immediately and quickly go to the next question. This hindered her ability to determine if the learners were willing to persevere in their struggle to solve difficult problems (Von Glasersfeld, 1987). From an interactionist perspective, the teacher limiting the discussion would also limit the development of conceptual understanding (Jaworski, 1994).

36. T: Can these two 3D objects be used for the same purpose in real life?

37. Learners: No ma'am

38. T: Why?

39. L: The prism is like this box (pointing at the box of books). We can put tins of fish and tins of jam.

40. T: What about the pyramid? Where do we find pyramids?

41. L: mh…. Roof of the house

42: T: Ok, why the roof?

43. L: because it is sharp ma'am.

44. T: but the roof of the house is not sharp, it's mmmh… okay look at this (pointing at a roof of a model house and comparing it with a pyramid block). What do you see? What is the difference between the two?

45. L: the roof of this house has the faces that do not go to one corner, this one (pointing at the pyramid) have faces that go to one corner.

46. T: oh! That's good; you see the roof is not a pyramid. This is a pyramid; you see this point is sharp. Now let us look at the prism. What do you see? Look at the sides, do you see something?

The teacher focused on 'why' in utterance 38 but sometimes missed some great opportunities that she could have used to get more reasoning from the learners. This resonates with what Schifter (2001) states that while the teacher may enter the classroom with a stronger mathematics background, there are additional mathematical skills that s/he needs to call on in order to respond to the learners' thinking, skills that are unlikely to be cultivated in explorations of mathematical content. In utterance 44 the teacher probed the learners to explore the characteristics of a pyramid however in utterance 46 she decides to tell the learners instead of following up with more probing questions. This deprived the learners of a chance to conjecture, present their arguments and to prove their conclusions. Carpenter et al.

(2003) also support the importance of conjecturing by stating that they have "found that it is productive to ask children whether their conjectures are always true and how they know they are true" (p102).

47. L: the sides of the prism has rectangles

48. T: and the pyramid?

49. L: has triangles

50. T: okay, you see the triangles in the pyramids and the rectangles in the prisms?

50. Learners: yes ma'am

51. T: I will give you homework and see if you really understand the difference between the prism and the pyramid.

52. L: yes ma'am

While there was little evidence of explanation of solutions in this class there was also no evidence of conjecturing and presentation of arguments. In utterance 50 the teacher quickly goes to homework without ensuring that the learners really understood the differences between prisms and pyramids. Moschkovich (2002, p. 193) indicates that "in many classrooms teachers are incorporating many forms of mathematical communication and students are expected to participate in a variety of oral and written practices such as explaining solution processes, describing conjectures, proving conclusions and presenting arguments" to enable understanding. Learners in this class had little time to share ideas with the teacher facilitating. Challenging the learners' ability to comprehend mathematical concepts, as well as encouraging learners to express ideas with peers was minimal. She could have encouraged interaction by probing the leaner to explain the difference between a square and a rectangle on her own. Ball and Bass (2003) insist that to ensure fruitful interaction in a mathematics classroom; teachers have to have the skill to probe learners to prove their answers so that he may understand the reasoning behind their explanations and justifications of answers.

CONCLUSIONS

Promoting mathematical reasoning is a challenge in the teaching of mathematics. The teacher selected a task that had the potential to promote mathematical reasoning, that is, both lower-level demand questions and higher-level demand questions. The task had the potential to develop the five strands of mathematical proficiency (Kilpatrick et al., 2001). The implementation however focused mainly on procedural fluency where the learners had to reproduce learned facts. This indicates that a good task in and of itself does not guarantee that the interactions that follow will promote mathematical reasoning. The teacher was restricted in her use of probing questions that would foster learner discourse and interaction, that would have lead to a greater demand for mathematical reasoning and hence developing mathematical proficiency.

This made it difficult for the teacher to facilitate the learners' participation in extended discussions, articulating their ideas and interacting with others' ideas to engage deeper mathematical thinking. There was also not evidence of the learners asking questions of the teacher during the lesson.

RECOMMENDATIONS

This research has shown that it is not sufficient for a teacher to select an appropriate task but the manner in which the task is implemented is also critical. Schiffer (2001) argues that, before the teacher gives the task to the learners he/she should consider the following:

- How he/she can attend to the mathematics in what the learners will be saying and doing?
- How he/she will assess the mathematical validity of learners' ideas?
- Have the skill to be able to listen to the sense in learners' mathematical thinking even when something is amiss?
- Have ability to identify the conceptual issues the learners are working on?

Schiffer (2001) argues that although a teacher may have a strong mathematics background she needs to be able to listen to learners with a sharpened curiosity and interest, then ask questions of the learners that mediate the learning process. (Schiffer, 2001). Davies (1997) suggests that teachers should listen for the promotion of mathematical proficiency which is more than just attending to answers in different ways, as this still leads to listening for a particular response. When promoting mathematical proficiency the teacher has to develop hermeneutic listening skills (Davies,1997). She should be able to listen in order to encourage participation and interaction between learners. Their listening should not make them direct the learners to some pre-given understanding but must show a willingness to interrogate the learners in order to get to the reasoning behind their responses. As Kilpatrick et al. (2001) argue the teacher should also let the learners give details when using and explaining their strategies and that the more the learners interact about mathematical ideas and concepts, the more they develop their mathematical proficiency.

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INTEGRATING TECHNOLOGY IN TEACHING INTEGRAL CALCULUS

Marguerite Walton

Lecturing mathematics at the Nelson Mandela Metropolitan University (NMMU) has traditionally been in a "talk and chalk" paradigm. With the increasing availability of various forms of technology more lecturers shift away from traditional lecturing methods to incorporating technology in lectures. This paper will discuss the incorporation of various forms of technology in the student support and lecturing of a first year integral calculus course. The effectiveness of the use of technology and the students' responses to the technology will be discussed.

INTRODUCTION

In the last decade there has been a global trend towards equipping lecture venues in higher education institutions with technology to use for education. During the last couple of years South African institutions have joined the trend as lecturers and students are becoming more and more technologically literate.

First year Integral Calculus at NMMU focuses on techniques of integration and applications of integration. Added to these topics, students are introduced to parametric equations and polar coordinates, first order differential equations and multivariable calculus. The purpose of this study is to present the variety of technological tools that were used to assist the students and to evaluate the effectiveness of these tools.

The lecturer used a Tablet PC to present the lectures. A Tablet PC is a type of notebook computer with a screen on which the user can write using a stylus pen. The writing is then digitized and can form part of the document (Weitz, Wachsmuth, & Mirliss, 2006). In the Integral Calculus course the lecturer used PowerPoint lecture slides with space available for ink annotations. The slides contained a framework for the lecture while during the lecture the mathematics was developed using the ink annotations.

All the students registered for the course were given access to a Module Site created on the local intranet which they could access from any computer on campus. This Module Site was used to disseminate information to the students. It contained initial documents handed out to students, namely the course outline and information sheet, information about test dates and venues, and other announcements that students might have missed during the lectures. In addition the Module Site was used to store documents that the students could need, namely previous years' test and additional exercises developed for students. The Module Site had a place for Web Links to helpful websites that the students could use for further explanations on topics. The Module Site was also used to store all the PowerPoint lectures with the ink annotations.

During lectures the lecturer used a graphing package, Autograph ®, to enhance the lectures further. The graphing package was used extensively during the sections on parametric equations and polar coordinates to assist with the explanations.

BACKGROUND

Many mathematics lecturers say that they do not have enough time during their lectures to cover the material contained in the module, yet they would argue that all the topics included are essential (Gorgievski, Stroud, Truxaw & DeFranco, 2005). Lecturers are using technological tools to assist them to cover the material in a time efficient manner (Mantei, 2000).

Even though technology could be an effective tool to assist lecturers, it is essential that the mathematical content and pedagogy is not compromised (Laurillard, 1993). Wenglinsky (2005) suggests that a lesson should not be planned around a computer, but a computer should be used to enhance a lesson. Garofalo, Drier, Harper, Timmerman & Shockey (2000) argue that technology should be used to enhance what could be done without technology. The technology should be used in context, should be worthwhile and should connect mathematical topics. Li (2003) warns that technology might make complex topics seem simpler, and even though technology plays an important role in the classroom, it is a classroom tool, a resource to the lecturer, no more.

Some studies have been done on students' perceptions of technology use, and the general feeling is that adding technology to courses that do not currently use technology has a positive impact on students' perceptions of the instructor and the course (Apperson, Laws & Scepansky, 2006; Lavin, Korte & Davies, 2009; Li , 2003).

In addition, Lavin et al (2009) found that technology has a positive impact on student preparation for class, attentiveness, quality of notes taken, student participation in class and student learning. Apperson et al (2006) found that students feel that technology enhances their learning, even though there was no difference in grades of students who were in lectures where PowerPoint was used compared to the lectures where PowerPoint was not used. On the other hand, Mantei (2000) found that the use of Internet notes and PowerPoint presentations improved student performance in examinations.

Focusing on PowerPoint as a technological tool, students feel that a lecturer using PowerPoint seems more organised, and the lecture presentation is clearer than lectures where PowerPoint is not used (Apperson et al, 2006). In addition, students felt that their interest was kept longer. The main benefit of using inking on the

PowerPoint slides is the ability to spontaneously provide diagrams and to actively work through examples instead of having pre-scripted PowerPoint slides (Weitz et al, 2006).

Using graphing tools in a lecture increases the students' interest level (Apperson et al, 2006). It is time consuming to sketch a graph that is used solely for explaining a concept and using a graphing tool can save time and give the students a visually effective link to enhance their understanding.

Mantei (2000) found that apart from being a time saving tool, Internet notes enhance student learning more than conventional presentation methods.

RESULTS

The students registered for Integral Calculus at NMMU are divided into two groups, namely prospective engineers and all the other students. This study will focus on the group of engineering students. A total of 28 questionnaires were completed (see Addendum 1 for the questionnaire).

Using the Tablet PC

PowerPoint slides were developed to be used in the Integral Calculus course. The slides had most of the theory (definitions and some theorems) typed on. The examples that were covered in class were typed on the slides without solutions. The lecturer used the lecture to discuss the theory and go through the examples. The students were able to see how the examples developed mathematically since the lecturer used inking on the slides to cover the example step-by-step. (See Addendum 2 for examples of two slides after the inking.)

The PowerPoint slides, with inking, were then placed on the Module Site for the students to download and/or print if they wanted to.

A benefit for the lecturer of using PowerPoint slides was that theory did not have to be written out on the board, which is time-consuming. It was rather discussed while referring to the PowerPoint slide. A whiteboard was available in the classroom to use for additional explanations if it was needed. A benefit for the students was that they did not have to write down everything that was on the board. They could rather focus on understanding and "fill in the gaps" in their notes later.

From the questionnaire, all the students indicated that they could easily read the inking on the TabletPC. All students preferred inking to the whole lecture typed up before hand, while only 4% indicated that they would prefer the lecturer to use a whiteboard/chalkboard.

Some of the comments students had on the Tablet PC were:

- It is not time consuming.

- Tablet PC is a good idea as the notes can be sent/put onto the module site and a student can later revise the notes from class.
- The writing of ink on the PC is as the same as writing on the board, but more efficient and advantageous because you get to review the notes when it is posted again.
- I'm okay with the Tablet PC because I can easily follow the worked examples which we do in class.
- The Tablet PC seems easier and everybody can see, wherein in chalkboard some people couldn't see at the back.
- It's a great idea and tool. I think it's a big help when learning and more lecturers should start making use of it.
- Find it helpful when working with the lecturer on a problem, following steps.
- When using the inking it gave me enough time as well to make my own notes and write down examples as the lecturer does them.
- Saves a little time (easy access to tables etc). By writing on the Tablet it is easier to process the information than if it was already typed up. (More time to think).
- Easier access to lecture notes as if lecturer writes on a white/chalkboard it can't be placed on the student portal.
- Simply put, it is faster than a chalkboard with less mess.
- The writing space on the Tablet PC is small so it give us minimal space to write.
- One improvement could be using a thinner 'ink' to make it more legible, and keeping it spaced.

The Module Site

All Integral Calculus students had access to the Module Site. The Module Site is the main tool used for dissemination of information to the students. There are enough computer labs available on campus for students to have access to the Module Site.

Of the students in the engineering group, 46% stated that they used the Module Site daily, while 46% stated they used it either once or twice a week. One student only used it monthly while one student said they only used the Module Site a couple of times during the semester.

93% of the students found the lecture notes placed on the Module Site helpful, while 100% of the students found previous tests and examinations helpful.

Some of the comments students had on the Module Site were:

- The module site for MATH103 was extremely helpful. It give insight into the module and was always stocked up with info.
- Access to assessments, past papers and exams were very helpful, for preparation for tests and exams.
- I found everything I needed ready any time I accessed it.

- Very colourful, I liked that there were questions to do, it really helped especially for the exams.
- The module site was an easy way to get our notes from class.
- The module site is perfect. Nothing needs to change. The notes are always there if we need it.

Using a graphing package during lectures

Parametric equations is a topic that the first years students generally enjoy. Once they get an understanding of how a graph develops as a parameter increases through an interval, the calculus on parametric equations follows more easily. Using Autograph ®, it was possible to show the students a variety of parametric curves. The graphing package has a "slow plot" function, so it was used to assist the students in visualising the development of the graph as the parameter increases. Students were required to sketch simpler parametric curves, but once the calculus on parametric curves (tangents, concavity, areas and arc length) was discussed, it was very handy to have a sketch of the relevant curve available. The use of the graphing package was not restricted to the section on parametric curves, although this was the section where it was most beneficial.

A web link to a 30-day free trial period of Autograph ® was placed on the Module Site. Students were encouraged to use it in their own time to assist them. Time was not spent in the lecture explaining how the graphing package worked, since it was not compulsory for the students to use it. In future, it will probably be beneficial to students to include a session on how to use a graphing package in one of the lectures.

96% of the students found the use of the graphing package useful and would have preferred it if it was used more in the lectures. 26% of the students indicated that they used a graphing package to assist them in their studying.

Some of the comments students had on the graphing package were:

- Autograph makes graphing more easier because it can be able to create 3 dimension graphs that cannot be created by hand drawing.
- Gives a clear image of how the graphs are formed at each stage.
- We didn't do much autograph but it did illustrate graphs in a helpful way.
- It's difficult to use and understand but it can be helpful if we are taught how to use it.
- Made it easier to visualise the curves.
- Much more accurate/neater than hand-drawn graphs. Personal comments on graphs can then be added.
- Helps us see and understand the graphs better and faster.
- Autograph is a good thing cause it helps you to visualize what is happening and have an idea of what you are doing.

- It helped me see how the polar curves were drawn degree by degree.
- Maybe a little more explanation could be made on how to use the package, to make it easier for students to use it while studying.

FINAL REMARKS

The Integral Calculus lectures were not dominated by technology, yet technology played an important role in saving time, disseminating information and increasing students' understanding of many topics. Both the lecturer and the students benefitted from the various forms of technology used.

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ADDENDUM 1: STUDENT QUESTIONNAIRE

Use of Technology in MATH103
Cross the chosen option

<u>TabletPC</u>

I prefer it if the lecturer writes on the TabletPC rather than whiteboard/chalkboard.	Yes	No
I can easily read the "inking" on the TabletPC.	Yes	No
I would prefer if the whole lecture is typed up, and no "inking" takes place.	Yes	No

Comments on the TabletPC:

<u>Module Site</u>
I accessed the MATH103 Module Site

Daily	**Twice a week**	**Weekly**	**Every second week**	**Monthly**	**Only a couple of times**	**Never**

I found the lecture notes helpful.	Yes	No
I found the previous tests and Exams helpful.	Yes	No

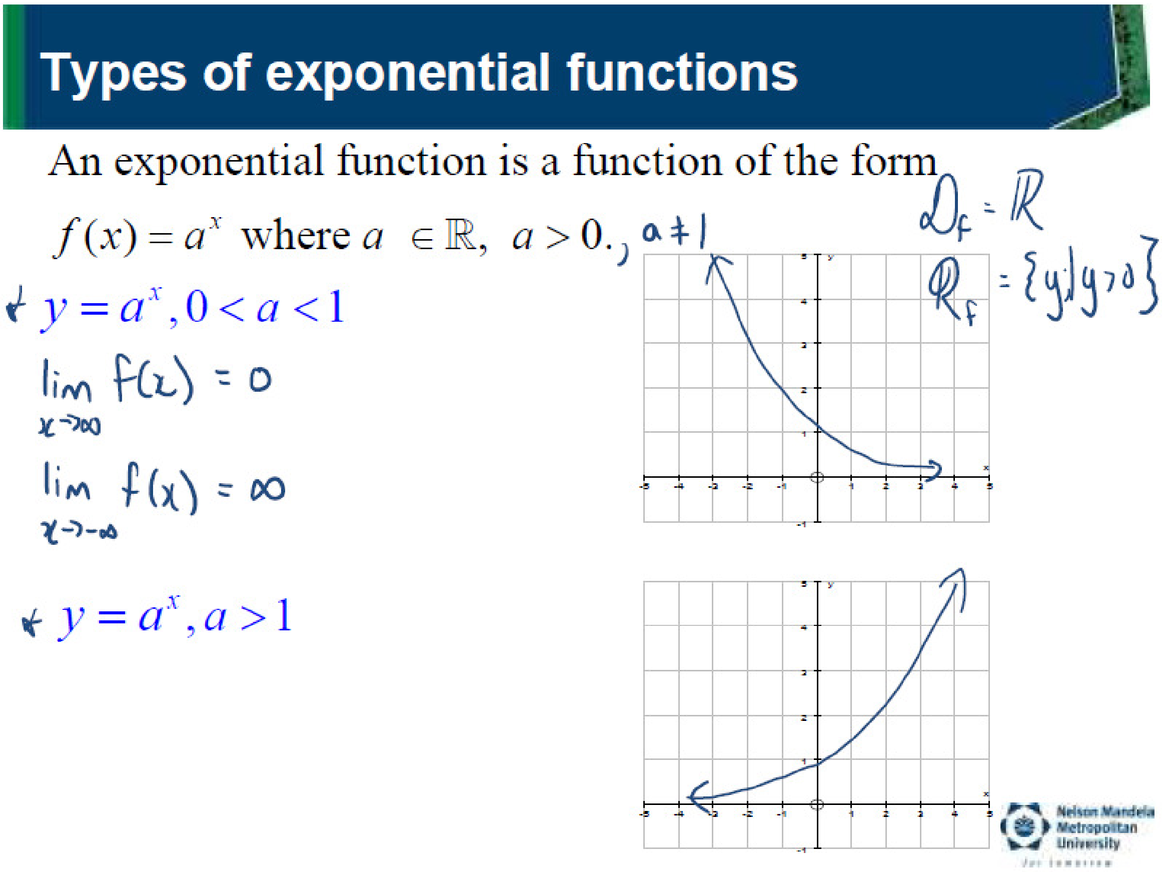
Comments on Module Site:

<u>Autograph</u>

I found the use of the graphing package in class useful.	Yes	No
I would like the lecturer to use it	More	Less
I used a similar graphing package in my own time while studying.	Yes	No

Comments on Autograph:

ADDENDUM 2: POWER POINT SLIDES WITH INKING



Exercise 20, p502

$$\int \cos^2(x)\sin(2x)\,dx$$

$$= \int \cos^2(x)\cdot 2\sin(x)\cos(x)\,dx$$

$$= 2\int \cos^3(x)\cdot \sin(x)\,dx$$

Let $u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -2\int u^3\,du$$

$$= \frac{-2u^4}{4} + C = \frac{-2\cos^4(x)}{4} + C$$

Nelson Mandela Metropolitan University

CHOOSING EXAMPLES FOR TEACHING MATHEMATICS - A 'KNOTTY' EXERCISE

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This paper explores the complexity of choosing examples for use in the mathematics classroom. The purpose of the paper is to create awareness and to provide a set of discursive resources with which to talk about examples and their use in practice. The paper makes use of extracts from a textbook that provides an empirical base which illuminates how examples can bring the object of learning into focus. The theoretical lens that is used to engage with the textbook is situated firmly in variation theory.

INTRODUCTION

This paper stems from my inherent interest to explore how a deliberate use of examples could support and enhance teachers' mediation of a particular object of learning. This interest is sparked by the fact that we learn through exemplification. In mathematics the use of examples is particularly important because mathematics is an abstract science and the only way for it to be illuminated is through exemplification. Mathematics textbooks across all levels are littered with examples - have you ever considered why? In recent research there has been a much more deliberate focus on examples in mathematics. To this end, a special issue of the journal 'Educational Studies in Mathematics' was published in 2008 (volume 69) that was dedicated to this topic. So the idea of the significance of the use of examples in mathematics has recently gained currency.

In planning a lesson what considerations should a teacher take into account when selecting examples to be used to illustrate a concept? What examples are used to reinforce these ideas? And what examples are given to learners for practice purposes at home? Does the choice of examples really matter? Is it sufficient to merely follow the example sequence as captured in particular textbooks to illustrate the concept to be taught? Is it productive for learners if we arbitrarily choose the questions from an exercise set in a textbook to engage learners with? I pose these questions to commence a possible process of self reflection on one’s practice as a teacher of mathematics so as to establish some critical thinking as to 'why am I choosing this specific example or set of examples'?

Let us pause for a brief moment and consider the following two examples:

Example 1:	**Example 2**:
Find the slope of the lines joining the	Find the slope of the lines joining the

following pairs of points:	following pairs of points:
a) (1;-3) and (-1;3) b) (4;2) and (6;2)	a) (-4;-7) and (2;5) b) (0;-2) and (4;6)
c) (2;-2) and (5;2) d) (-1;-1) and (-4;8)	c) (4;2) and (0;0) d) (-2;0) and (4;3)

In both examples learners are required to determine the gradient of the line passing through the given set of points. Which example set would you choose for your learners and why? Knowing the difference between the two example sets and being able to identify the possible space of learning each of the example sets could open is crucial knowledge for a teacher to be able to select from the two.

The purpose of this paper is to create awareness and to provide a set of discursive resources about mathematical examples and their use in practice. The focus of the paper is to demonstrate how examples can be used to illuminate an object of learning. To accomplish this task I will make use of examples from one textbook, viz. FOCUS on mathematics - grade 10, to assist in illuminating the *'knotty'* exercise example selection poses. This particular textbook was selected because it is a textbook of my preference. The purpose is not to critique or analyse the textbook but merely to use it as empirical evidence for this paper. I will start by providing a brief overview of the theory that informs my discussion.

Laying the Foundation - Learning as Experiencing Variation

Ference Marton and his colleagues[34] have developed a theory for learning which they have called ‘variation theory’. The central tenet of this theory lies precisely in the meaning of the word variation i.e. we learn through the experience of difference, rather than the recognition of similarity. Marton and Pang elaborate that in order to "learn something, the learner must discern what is to be learned (the object of learning)” (Marton & Pang, 2006, p. 193). In Runesson’s words “to be able to see what is the case, I must see what is *not* the case” (Runesson, 1999, p. 2). One can therefore infer that the experience of *variation* is a necessary condition for *discernment*. Thus, the kind of variation embedded in the set of examples a teacher makes use of during his/her teaching is crucial in the teacher’s attempt to keep the object of learning in focus during the course of a lesson.

Bowden and Marton (1998) as cited in Runesson (1999, p. 3) argue, that “when some aspect of a phenomenon or an event vary while another aspect or other aspects remain invariant, the varying aspect will be discerned.” In view of this, the variation that learners are exposed to must be important for their learning. Mason and Watson (2005) argue that an inescapable theme in mathematics is *invariance* in the midst of change. Thus, another way in which the ability to discern can be influenced, is to

[34] See (Marton & Booth, 1997); (Runesson, 1999); (Marton & Pang, 2006)

experience invariance against the backdrop of change. They further argue that if learners are to experience variation, and if they are to learn from it then there must be sufficient variation and in sufficiently quick succession for it to come to their attention. Mason and Watson (ibid.) also indicate that to experience variation, means more than simply being exposed to it. It needs to be a repeated experience with ‘emotional as well as cognitive and affective engagement’. It needs to be repeated until some degree of familiarity develops.

The capacity of a teacher to engage his/her learners with tasks, exercises or examples that would allow them to experience variation is dependent on the teachers’ knowledge and understanding of the concept in question. As Mason and Watson put it “… it is of major advantage to learners if their teachers are themselves aware of the various dimensions of possible variation” (Mason & Watson, 2005, p. 5). Of specific importance to the discussion of this paper is how we use/select examples to bring the object of learning into focus.

SETTING THE SCENE

Chapter 9 from the FOCUS on mathematics - grade 10 textbook deals with graphs of linear equations[35] and starts by investigating the properties of graphs with the equation $y=mx+c$. Two examples are 'discussed' which implicitly deal with different representations of the functions $y=2x-3$ and $y=\frac{-1}{2}x+1$ (these being an algebraic equation, a table of values and a graphical representation). The focus in these examples is on the gradient and it is used to demonstrate the notion of an increasing and decreasing graph. The actual investigation of the properties of the graph $y=mx+c$ is only dealt with in the first exercise of the chapter (exercise 9.1), questions 8 and 9, they are:

[35] Note that in the tile the term 'linear function' was not used. It is also not used throughout the chapter, instead reference is made to expressions such as 'graphs with the equation $y=mx+c$' or 'straight line graphs' or 'linear equation'. I am inferring that this was a deliberate move by the author since the use of the term 'linear function' would seduce learners into thinking that any line is a function since we talk about 'linear functions'. They would therefore be able to correctly identify the line defined by $y=x+2$, for example, as a function without having any understanding of the concept of function. As a result of this, they would probably also refer to the line defined by $x=3$ for example as being a function as well.

8. Consider the graph with equation $y = 2x$. Form a mental picture of the graph. Do not draw it! Tell your partner what is happening to the graph of $y = 2x + c$, if:
(a) $c = 1$ (b) $c = 5$ (c) $c = -3$ (d) $c = -4\frac{1}{2}$
Ask your partner to draw the graphs as you describe them.

9. Consider the graph with equation $y = x + 1$. Form a mental picture of the graph. Do not draw it! Tell your partner what is happening to the graph of $y = mx + 1$ if:
(a) $m = \frac{1}{2}$ (b) $m = \frac{1}{4}$ (c) $m = 2$ (d) $m = 3$
(e) $m = 4$ (f) $m = -4$ (g) $m = -3$ (h) $m = -2$
(i) $m = -1$ (j) $m = -\frac{1}{2}$ (k) $m = 0$
Ask your partner to draw the graphs as you describe them.

(Bennie, 2009, p. 134)

Firstly, why has Bennie (ibid.) provided so many examples in question 9 as compared to question 8? Which examples from question 9 would you choose to give to your learners and why would you make this selection? With respect to question 9 what we see is that the y-intercept remains constant while the gradient varies. So in terms of variation theory this question provides an opportunity for learners to discern the effect of the gradient on the straight line. However, I wonder, given the particular sequencing of the change in gradient would its effect be easily discerned by the learners. The current sequence starts with the gradient decreasing in value (questions a to b), then it increases (questions c to e), it moves back to decreasing values (questions e to f) and then starts to increase again (questions f to k). In this question we have the gradient varying but at the same time the sequencing within the question provides another level of variation that could be distracting for the learners and this needs careful consideration.

I think that if this question made use of larger negative and larger positive values of m, and if the questions were arranged in such a way that the gradients were either increasing or decreasing from questions a to k, then the opportunity for learners to discern the effect of the gradient on the straight line would be strengthened. Given the particular values of m in question 9, if we took values of m in increasing order then learners would be able to see that as m increases from -4 to $-\frac{1}{2}$, reading the graph from *left to right*[36] the lines *slope down to the right* and the line becomes *flatter.* With m=0 learners will notice that we are now dealing with the case of a horizontal line. So continuing with increasing values of m from $\frac{1}{4}$ to 4, learners will be able to observe that the lines now *slope up to the right* and become *steeper*.

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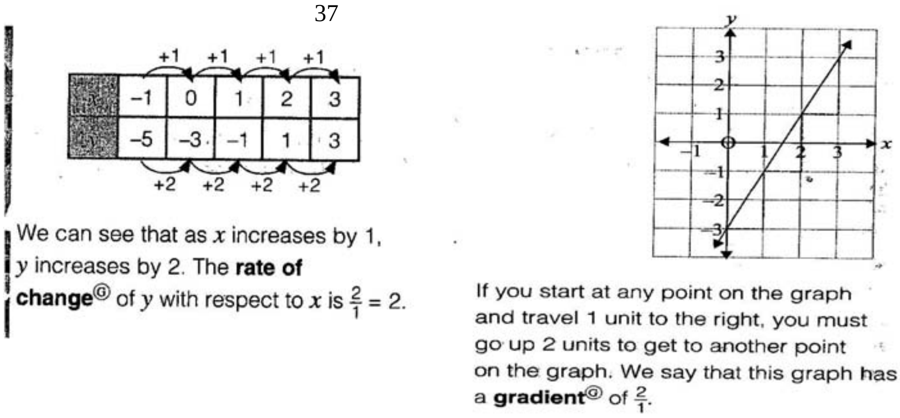
[36] The language used to describe the slope of the straight line (all italicised) is taken from the work done by the Wits Maths CONNECT project for secondary schools that dealt with the gradient if the linear function with grade 9 learners.

OPENING THE SPACE FOR LEARNING

I started the previous section by describing the way in which the author introduces chapter 9. I indicated that the author uses two examples that implicitly deal with different representations of the functions $y = 2x - 3$ and $y = \frac{-1}{2}x + 1$ viz. an algebraic equation, a table of values and a graphical representation. The author uses these representations to bring the notion of gradient into focus:

[37]



(Bennie, 2009, p. 131)

Within this task there lies a hidden pedagogical possibility. Gaining access to this possibility is dependent on the dimension of variation that is opened (Marton, Runesson, & Tsui, 2004). In this instance, consider the different representations as variations of the form in which the straight line can exist - as an algebraic equation or a table of values or a graph. The fourth representation, viz. verbal/words (which maybe cumbersome to capture in a textbook) - words that describe the process in terms of input and output values. For instance $y = 2x - 3$ means double the input value then subtract 3 to get an output value.

Now, each of these representations describes how the value of one variable is determined by the value of another. Teaching for what Kilpatrick, Swafford and Findell (2001) refer to as mathematical proficiency implies that learners need to understand that there are different ways of describing the same relationship. This does not only mean developing learners' capacity to perform various procedures such as finding the value of y given the x value or creating a graph given an equation, but should also include assisting learners in developing a conceptual understanding of the function concept. "A significant indicator of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes" Kilpatrick *et al.* (2001, p. 119). Going back to the example and opening the dimension of variation in terms of the

[37] The superscript © means refer to the glossary section of the textbook.

different representations, what is the possible space of learning that can be created?

These different representations are normally taught as processes that one needs to follow to move from an equation to the graphical representation where the table of values acts as the intermediary. The space of learning which is then opened is an unconnected space i.e. there is no meaningful link made between the different representations and they are all seen as a means to an end. In the example, the author shows the manifestation of the gradient in each of the representations. Now consider asking learners to show the y-intercept in each of the representations. Using this task we could ask learners to explain what happens in each of the representations if the y-intercept becomes 5, then 7, then 12 and so on. What about also asking learners to show what happens in each of the representations if the gradient changes. I think that by engaging learners with this kind of activity we begin to provide learners with opportunities to engage with a function as a correspondence between two sets - a set of possible inputs to the process and a set of possible outputs from the process. It is in opportunities like this that the learners can begin to reason about functions in what Thompson (1994) refers to as functions as ***objects*** or what Sfard (1992) refers to as a structural approach to mathematics. When learners begin to reason about function as an object they are able to "conceive of the entire process as happening to all values at once, and is able to conceptually run through a continuum of input values while attending to the resulting impact on output values" (Oehrtman, Carlson, & Thompson, 2008).

The responsibility of the teacher when teaching is to open up an appropriate space of learning for his/her learners and this space is opened through the teachers' selection of examples. To accomplish this task the teacher needs to fully understand what the object of learning is and then in relation to this the teacher needs to know what the possible dimensions of variation are and the range of permissible change. Going back to question 9 from exercise 9.1 as cited earlier, if I included m=100 to the example sequence - what would this wider space of variation allow learners to experience? How would this specific example impact on the space of learning? It certainly brings in the need to work with scale, thus increasing the space of learning - but its relevance to the object of learning also needs to be considered.

In exercise 9.2, question 1 (Bennie, 2009, p. 136) learners are required to use either the dual intercept or gradient and y-intercept method, which is the focus of attention in the section leading to exercise 9.2, to draw the graphs defined - nine equations are thus given. In question 2 of this exercise, the learners are now expected to use the graphs drawn in question 1 and explain what they notice about the following pairs of graphs defined by:

1) $y=2x-4$ and $y=-\frac{1}{2}x+1$ 2) $y=\frac{3}{4}x-2$ and $y=-\frac{4}{3}x$

Should the learners be focusing on the perpendicular relationship between the given pairs of lines or their different y-intercepts? Consider the following modification to the equations:

1) $y=2x-4$ and $y=-\frac{1}{2}x-4$ 2) $y=\frac{3}{4}x$ and $y=-\frac{4}{3}x$, now what aspect is brought into focus?

CONCLUSION

Watson and Mason (2002) indicate that teachers frequently use examples in order to demonstrate and communicate the essence of mathematical concepts and techniques. This is amplified by Zazkis and Chernoff's (2008) discussion that examples are an important component of expert knowledge and that examples are used to verify statements, to illustrate algorithms and procedures, and to provide specific cases that fit the requirements of the definition under discussion. So the use of examples forms a crucial part in the teaching and learning of mathematics.

Being exposed to the essential features inherent in variation theory and coming to realise the important role examples play in the teaching of mathematics, useful questions that I ask myself when selecting examples for use in my practice are: i) What is the intended object of learning for the lesson? ii) What are the possible dimensions of variation with respect to the object of learning? iii) What is the range of permissible change? iv) What is varying and what is invariant in the example set? v) Does the example set provide opportunities to open an appropriate space of learning? vi) If required, how can I modify the example set in a textbook or other resource so as to ensure that when given to learners they provide opportunities for learners to discern the object of learning?

The purpose of this paper is to highlight what the 'knotty' exercise example selection poses when considering the use of examples in teaching and to provide some discursive resources that allow us to talk about examples in use, in a systematic and developmental way. As illustrated in this paper, the *space of learning* that is opened by the examples a teacher uses is dependent on the possible *dimensions of variation* that the teacher is aware of. This is then constrained by knowing what *the range of permissible change* is. Marton, Runesson and Tsui (2004) stress that providing opportunities and conditions for learning does not in itself cause learning to take place. They argue that when the object of learning has gone out of focus, it needs to be brought back into focus. But it is not done in and of itself; it is done in relation to the learner. Thus, Marton *et al.* (2004, p. 32) argue that "the kinds of examples and analogies the teacher uses, the stories that the teacher tells, the contexts that the teacher brings in and so on" are important for the constitution of the space of learning.

Acknowledgement

This paper arises from work that is funded by the Sasol Inzalo Foundation and the

National Research Foundation (grant number 71218). Any opinions, findings and conclusions or recommendations expressed in this paper are those of the author and do not reflect the views of either of the foundations.

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ARE OUTREACH PROGRAMMES IN MATHEMATICS AND SCIENCE A NECESSITY? SOME PERSONAL REFLECTIONS!

VG GOVENDER

NELSON MANDELA METROPOLITAN UNIVERSITY

This paper traces the involvement of the writer in outreach programmes for mathematics and science learners. The writer has gained considerable experience and insight into such programmes. Two programmes are described in detail and the roles played by the writer in them are outlined. Data from these programmes suggest great benefits for learners involved in them. The writer uses this data and personal experiences to attempt an answer to the question posed in the title of this paper.

INTRODUCTION

In South Africa, concern about achievement levels in mathematics and science among African students has led various stakeholders such as the private sector, the department of education, non-governmental organisations and higher education institutions to provide supplementary tuition to increase equity in educational outcomes among high school students (Reddy, Berkowitz and Mji, 2006). Supplementary tuition has usually been organised through a number of different outreach programmes for learners. I have been privileged to be involved in a number of outreach programmes for learners. My involvement has been very rewarding as I have gained considerable knowledge and expertise about planning and implementing such programmes. It has helped the learners in the programmes to improve or excel in Mathematics and Physical Science and enhanced their prospects of gaining access to higher education.

WHAT ARE OUTREACH PROGRAMMES?

There are a variety of outreach programmes for learners. The sponsors of outreach programmes usually prescribe the target group. Billie E. J. Housego (1999) speaks about an outreach programme in Alberta, Canada where “teachers in innovative, alternative schools and programs have attempted, for more than 30 years, to meet the needs of students who either cannot or will not pursue their education in traditional high schools”. In South Africa, it would appear, through a variety of reasons, that certain high schools, usually in less affluent areas, are not able to give learners quality instruction that is expected in subjects such as Mathematics and Physical Science. Learners from these and other schools are always looking for extra classes where supplementary tuition is provided. Mogari, Coetzee and Maritz (2009) report on research by Ireson and others which show there is a strong relationship between learners’ opportunity to participate in supplementary tuition and their socio-economic background. Lauziere (2010) speaks about American parents waiting until there is a crisis at school before seeking help in mathematics and science. This is probably true for South African parents as well. However, in the South African context, only the middle and upper classes can afford supplementary tuition by a private tutor. The prohibitive cost of private tuition puts it out of reach for many South African households, hence the need for funded outreach programmes.

MY INVOLVEMENT IN OUTREACH PROGRAMMES

In this paper I reveal details about my involvement in outreach programmes from my personal experiences. Personal experience papers are usually the most fulfilling to write because they describe a significant event in one's life (Thompson, 2007). However, personal experience papers may appear to be subjective with issues of reliability and validity surfacing. To overcome this "subjectivity", I attempt to combine fact with both quantitative and qualitative data, where appropriate.

I start with a summary of my involvement over the past 19 years. This is shown in table 1.

Programme	My role	Years	Target group	Materials
Ethembeni Enrichment centre	Mathematics tutor	1992 – 1995	Grade 12 learners from disadvantaged schools	Compiled by tutors
UPE Ripple Programme	Mathematics tutor	1993 – 2001	Grade 12 learners from disadvantaged schools	Compiled by tutors
CELL C GEMS Programme	Coordinator	2004 – 2006	Grade 10-12 girl learners from disadvantaged schools	Supplied by NGO, PROTEC
NMMU Incubator School Project	Mathematics facilitator; project leader	2007 – 2011	Grade 11- 12 learners from mostly disadvantaged schools	DVDs and hard copies of material

Table 1 Programme details

A BRIEF DESCRIPTION OF MY INVOLVEMENT IN THE PROGRAMMES

As indicated in table 4, I was involved in the first two programmes until 2001. After a few years of involvement with teacher support programmes, I was offered an opportunity to be in a new programme sponsored by CELL C, which targeted girl learners. It was in 2004 and I was a Subject Advisor for Mathematics at the PE District Office. I had a second job, that of Coordinator for Mathematics, Science and Technology Education (MSTE) in the district. I was given the task, as MSTE coordinator in the Port Elizabeth district, to head this new project, called ***GEMS (Girls excelling in Mathematics and Science).*** Port Elizabeth was the second site of this programme, having been implemented in the Orange Farm area of Gauteng, in 2003. I did all the initial administrative work, including arranging meetings with school principals, selecting learners, obtaining venues and appointing tutors. A group of 40 grade 10 girl learners were selected to be in this programme. They were selected by their schools on the basis of their potential in Mathematics and Physical Science. I appointed the tutors, based on input from colleagues and school principals. The tutors were also very experienced local teachers and worked very well in the three years of the project.

Classes were held in Mathematics, Physical Science, English, The World of Work and Technology. Materials were sourced from PROTEC in Gauteng. CELL C arranged for the training of the tutors in the first year of the programme. The project commenced in mid-2004 when the girls were in grade 10 and continued until 2006 when the girls were in grade 12. As coordinator of the GEMS programme, I always had to visit the classes and observe the interaction between the tutors and the girls. I was very pleased by what I saw. Tutors were well prepared and the girl learners were actively engaged during the lessons. Tutors used a mixed mode of lesson delivery: whole class teaching; cooperative learning, class discussion, experimental and practical work. The assessments were varied and included tests, assignments, oral presentations, play acting, group assessments and portfolio presentations

I also checked with tutors about those girls who were not performing well. I sent regular reports to CELL C as well as the schools. This programme was intended to support these girls and to add value to what they learnt at school. I asked the girls about what being involved in the CELL C GEMS programme meant to them and if they gained anything from the programme. The quotes below (from individual girls) answered this question.

> *“To work with other students and accept that we are all different”*
>
> *"That I am the only one who can improve my lifestyle and find a great job”*
>
> *"That I am unique and must have self-confidence”*
>
> *"It has taught me that education is important and working with other learners helps you get to experience new things everyday”*
>
> *"To share my knowledge with others”*
>
> *"To be responsible for my work”*
>
> *" How to work with other people from a different race”*

"I have learnt that I have creativity in me"

"I am able to communicate better"

"I have learnt to value education"

"I have improved in maths and science"

"It was wonderful to share information in our groups"

These powerful statements from the girls show that they gained far more than Mathematics and Physical Science knowledge. The programme also prepared them for the world outside. As stated earlier, each tutor gave the girls a number of assessments and the marks were recorded on a schedule. All learners received participation certificates and marks reports. Top learners also received special prizes from CELL C. Summaries of these results (for 2004 – 2006), are listed in the three tables that follow. These results were retrieved from old files.

SUBJECT	**NO PASSED**	**NO FAILED**	**% PASS**
ENGLISH	38	02	90
MATHEMATICS	28	12	70
PHY. SC	30	10	75
WORLD OF WORK	38	02	90
TECHNOLOGY	37	03	93
OVERALL	**32**	**08**	**80**

Table 2 CELL GEMS Results: 2004

Of the 40 girls, 39 passed their grade 10 final examinations. Due to transfers and other problems, 33 girls continued with the programme in 2005.

SUBJECT	**NO PASSED**	**NO FAILED**	**% PASS**
ENGLISH	33	-	100
MATHEMATICS	24	9	73
PHY. SC	32	1	97
WORLD OF	33	-	100

WORK			
TECHNOLOGY	33	-	100
OVERALL	**32**	**1**	**97**

Table 3 CELL GEMS Results: 2005

In 2006, Technology was discontinued in order to give more time to Mathematics and Physical Science as they were in grade 12 then. Although all 32 girls passed grade 11 at school, only 26 girls started and completed the programme in 2006. CELL C did not want any new girls to be added in 2006.

SUBJECT	**NO PASSED**	**NO FAILED**	**% PASS**
ENGLISH	26	-	100
MATHEMATICS	18	8	69
PHY. SC	24	2	92
WORLD OF WORK	26	-	100
OVERALL	**26**	**-**	**100**

Table 4 CELL GEMS Results: 2006

CELL C was very impressed with the "high standard" in the programme (personal communication: M. Stephen, CELL C Transformation manager, 2005). However, the programme came to an end in August 2006. While these internal results seemed to be good, the results in mathematics were a cause for concern. However, all 26 girls went on to pass their final grade 12 examinations and a number of them went on to study further, doing degrees in commerce, pharmacy and science among others. As coordinator of the programme, it was a tremendous pleasure for me to observe visible growth in the girls, from being very shy, reserved girls in 2004 to confident, outgoing girls in 2006.

The Nelson Mandela Metropolitan University (NMMU) ***Incubator School Project*** started in 2005 and due to my commitment to the CELL C GEMS programme, I was not able to play a role in the beginning stages of this project. I was, however, invited to be on the committee as a representative of the Department of Education. Once my obligations to the CELL C programme was completed, I was able to join the programme as a Mathematics facilitator in 2007. By 2008 I was actively involved in the management team of the project, as I had joined the University by then. The ***Incubator School Project*** was housed in a unit of the Mathematics Department of

NMMU, called the Govan Mbeki Mathematics Development unit (GMMDU). Learners for the project were selected from a wide range of schools in the Port Elizabeth and Uitenhage area, with more than 80% coming from disadvantaged schools. Selection was done on the basis of learner potential. Thus, learners with a minimum of 50% in Mathematics and Physical Science in the final grade 11 examination were usually considered for the project.

I was very fortunate that by then the programme had evolved to a very high tech programme, using the DVD model in a blended learning environment. It meant that as facilitator I had to go prepare for each lesson thoroughly by going over the DVDs and ensure that I used a mixed mode of delivery in the classroom. Each DVD consisted of a series of small lessons in mathematics. The DVDs were played to the learners, with pauses at appropriate sections of the DVD. This was to enable me as facilitator to give constructive input to learners and to facilitate discussion at key points in the lesson. Learners were also given tutorials to work through. Each learner was given a set of DVDs. They were encouraged to go over the DVDs and prepare for tests that followed. They were also asked to share their resources with other learners at their schools.

By 2009 my role in the project had evolved to quality assuror and project leader. My role as project leader became more complex in 2010 and 2011. I became responsible for marketing, selection of learners, moderation of tests, appointment of staff, organising launch and award functions and budgets. By this time, the DVD model become more entrenched and was used in other projects of the unit. The DVD model had also been extended to the Physical Science part of the programme. NMMU made available 20 bursaries per year (from 2009) to learners in the project, ensuring that they had access to higher education. These learners (who received bursaries) and other learners from the project are now studying at NMMU and other universities. The boxes that follow show the grade 12 final results of learners in the project for 2010 and 2009. These results have been obtained from schools by our office staff and have been reported upon at the launch and closing ceremonies of the ***Incubator School project***.

2010 Grade 12 Mathematics Profile

From the 106 learners

- 73 obtained 50% and above
- 49 obtained 60% and above
- Symbols obtained
  - 9 x A + 16 x B + 24 x C
  - 24 x D + 17 x E + 11 x F
  - 5 Failed

2010 Grade 12 Physical Science Profile

From the 106 learners

- 49 obtained 60% and above
- 77 obtained 50% and above
- Symbols obtained
  - 6 x A + 22 x B + 21 x C
  - 28 x D + 18 x E + 10 x F
  - 1 Failed

2009 Grade 12 Mathematics Profile

From the 113 learners

- 95 obtained 50% and above
- 71 obtained 60% and above
- Symbols obtained
  - 16 x A + 19 x B + 36 x C
  - 24 x D + 12 x E + 5 x F
  - 1 Failed

2009 Grade 12 Physical Science Profile

From 110 learners

- 97 obtained 40% and above
- 46 obtained 50% and above
- Symbols obtained
  - 1 x A + 2 x B + 14 x C
  - 34 x D + 41 x E + 15 x F
  - 3 Failed
  -

These results have been exceptional, making the ***Incubator School Project*** one of the key engagement projects of the university. I have listed possible reasons for this success below:

- The classes are structured and learners know what to expect, based on feedback from others who participated in the project.
- Facilitators are experienced, prepare very well and use a variety of methods when going over the lessons. This caters for different learning styles.
- Learners are also given a variety of resources, included DVDs and hardcopies of the lessons. They use these resources to revise and prepare for tests in the next contact session. This ensures that their revision is ongoing.
- Learners sacrifice their Saturdays and show a lot of discipline. They sit in class from 8:30 till 13:00 with one break at 10:30. This is almost like another school day.
- They realise that if they do well they will be accepted at University and are in line to get bursaries. This motivates them further.
- The social aspect is also very important. Learners socialise with learners from other schools, build new friendships and also learn from each other. I have observed that these friendships and associations continue when they get to university.

VIEWS OF THE SCHOOL COMMUNITY ON OUTREACH PROGRAMMES FOR MATHEMATICS AND SCIENCE

Having worked closely with schools over the years, I have observed that most schools are very keen to get involved in outreach or intervention programmes. I have been in conversation with school principals and teachers and they attach great significance to involvement in such programmes. When I became project leader of the ***Incubator School Project***, I conducted a brief survey of school principals and teachers whose schools have regularly been involved in outreach programmes. I also interviewed some learners. Since the ***Incubator Project*** is an ongoing project, some direct references are made to this project.

School Principals

- Outreach programmes are very helpful in assisting the teachers at school get better performances from the learners.
- Learners are exposed to new approaches of working with their subjects and it gives them a different insight to what has been done at school.
- Staff, learners and parents have a positive attitude to the programmes and are supportive.
- The programmes help to improve learner results and enable them to focus on their career paths.
- There should be close cooperation between the schools and the programmes, especially with regard to the scheduling of the work (this was done in the scheduling of the topics in the ***2011 Incubator School project***)

Teachers

- It gives learners, with potential, who cannot afford extra tuition, an opportunity to excel.
- It exposes learners to different teaching styles and methodologies
- It builds the confidence of learners and helps improve their marks. In this regard the ***Incubator School Project*** was mentioned as having a positive impact on learners' results, with marks increasing from grade 11 to grade 12 by as much as 20%.
- After the Saturday class, where learners have picked new knowledge and skills, they return to school with key questions for their teachers. These questions are discussed in class, resulting in benefits for all learners at the school.
- The use of technology and other support material in the Incubator School is very helpful to learners. Initially, learners were concerned with the use of technology in the ***Incubator School Project*** as they were not used it. However, once they got used to this new approach, learners became more positive and confident and their learning was greatly enhanced.

Learners

- Learners believe that outreach programmes touch on critical parts of the syllabus and enables learners to correct any mistakes or misconceptions they may have.
- They feel that being involved in such programmes will enable them to improve in key subjects such as Mathematics and Physical Science.
- They interact with learners from other schools, compete with them and also learn from them.
- Learners welcome the opportunity of being in other teachers' classes and learning new approaches to solving problems in Mathematics and Science.
- Learners appreciate the fact that key examination type questions are also discussed in the classes and this helps them when they prepare for their final school examinations

It is evident from the views expressed by the school principals, teachers and learners about the value of outreach programmes. All, especially the learners, gain significantly from being involved.

WHAT HAVE I LEARNT FROM MY INVOLVEMENT IN OUTREACH PROGRAMMES?

As stated earlier, I have found my involvement in outreach programmes to be very rewarding and I have experienced and learnt a lot. I outline some of these experiences and learning below:

- Most middle and upper class parents are able to afford private tuition for their children. Even though these children may go to the best schools, there is often the need to achieve even higher. I have personally tutored some of these children.
- At the same time, at a number of schools, especially in less affluent areas, learners get a raw deal. For learners at these schools their only hope is support from experienced, dedicated teachers, who they are not likely to find at their schools. For them, outreach programmes provide a lifeline. I have seen learners from these schools really prosper in outreach programmes and attain the heights they desire.
- The success of outreach programmes such as the ***Incubator School Project*** has drawn interest from learners in better performing schools in more affluent areas. We have moved to a more inclusive approach in learner selection and currently have about 15% of such learners in the project.
- I admire learners in outreach programmes because they sacrifice their Saturdays. While their friends are out doing what teenagers usually do, they are

busy writing tests and attending important classes in Mathematics and Physical Science. For these learners Saturday has become another school day. They do not complain as they know they are working for better results and a better future.

- The study material used is a key factor. I have spelt out in the various programmes that I have been involved what study materials or resources were used. This has a major impact on teaching and learning. Unfortunately, this is also related to the budget. Resources, such as those used in the ***Incubator School Project,*** can be very expensive.
- Regular assessment is another key factor in the success of outreach programmes. Learners are compelled to go over their work very thoroughly in preparation for their assessment tasks. In the ***Incubator School Project***, learners write weekly tests in Mathematics and Physical Science. These tests are moderated, are of a very high standard and are used to consolidate their learning.
- Outreach programmes must provide something different to the learners. These programmes should use methods which are different from what they are used to. This will keep learners interested and maximise their learning. In this regard, the choice of tutors or facilitators is crucial. They must be experienced, have a good track record at their own schools and be amenable to trying out new methodologies. The ***Incubator School Project*** is one where it is compulsory to use innovative methods.
- The majority of learners in outreach programmes come from less affluent areas. They are provided with expert tuition and do well. Due to socio-economic circumstances, they cannot afford higher education. Providing incentives such as bursaries or loans can help these learners achieve their dreams of further study.
- It is important that there is a close working relationship between the organisers of outreach programmes and the schools that are part of the project. It must be seen as both parties working together to uplift the standard of mathematics and science education. School principals and teachers must realise that if they fail to respond to invitations to select learners for outreach programmes, they are doing their learners a disservice.
- It is very important to celebrate success. In the ***CELL C GEMS*** programme and the ***Incubator School project***, all participating learners were/are given certificates of participation and the top learners (based on assessment/test marks) rewarded with special prizes at the awards functions. It is a way of celebrating their successes and motivating them for their final examinations.
- Although class size is a key part of any outreach programme, budgetary constraints prevent classes from getting too small. It is possible for classes in outreach programmes to be more than 50. In this regard, good facilitator preparation and judicious use of technology and other resources overcome the negative effects of large classes.

CONCLUSION

There have been a plethora of outreach programmes for mathematics and science in South Africa. My involvement in a number of programmes has given me an insight into the way our education system works and the challenges faced by the teachers and learners. Learners, especially from disadvantaged schools located in less affluent areas, are desperate for extra classes or supplementary tuition in Mathematics and Physical Science, to name just two subjects. Data from two of the programmes suggest that learners tend to improve their marks in Mathematics and Physical Science as a result of this involvement.

I use the insight gained, my personal experiences and the data collected from the programmes to provide an answer to the question, “Are outreach programmes in mathematics and science a necessity?” When I look at my journey through the various programmes over the years, I must answer this question with a resounding “***yes***”. While it is crucial that teaching and learning at all our schools be improved, supported and strengthened, as long as South Africans are mired in poverty and our schools do not deliver what our learners need, outreach programmes are here to stay. It is the only hope for many of our learners who have the potential to excel, but this potential is not being nurtured at their schools.

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