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On The 3D Incompressible Navier-stokes Flows Around a Rotating Obstacle

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Abstract

We consider the motion of an incompressible viscous fluid, governed by the well known Navier-Stokes system of equations, in an exterior domain $\Omega c R^3$ with a smooth boundary $\partial\Omega = \Gamma$, which is assumed to be infinitely differentiable. The open bounded domain itself is assumed to possess a cone property [1]. When the rotation of an obstacle is taken into account, for a 3D case only the existence of the weak solutions has been confirmed in [4]. In this paper, we proceed one step further and confirm the uniqueness of that weak solution, using the so-called "energy method".

Keywords: viscous flow, rotating obstacle.

1. Introduction

The setting of the problem is 3D viscous, incompressible Laminar flow around a smooth impenetrable regular obstruction with a fixed axis of rotation slightly away from the centre of the channel of flow. The result of fluid flow with an obstacle is that, we end up with a net rotation (at angular velocity ω) in a particular direction. When the fluid flow is in contact with the rotating obstacle, the normal component of the flow surface velocity, $\gamma_0\vec{u}\,\mathbf{n}$=0. All the other surface velocity components are tangential to boundary of the rotating obstacle.

2. Statement of the Problem

We look for the unique solution $\vec{u}(x,t) \in H^2(\Omega)\text{ x }[0,T), T < \infty,$ such that,

$$(1)\begin{cases}(a)\ \rho\dfrac{\partial\vec{u}}{\partial t}+\rho(\nabla.\vec{u})\vec{u}=-\nabla p+\mu\Delta\vec{u}+(\vec{\omega}\Lambda\vec{x}).\nabla\vec{u}+\rho f\\ \quad\text{Subject to :}\\ (b)\ \vec{u}_{\partial\Omega}=\vec{\omega}\Lambda\vec{y};\\ (c)\ \nabla.\vec{u}=0;\\ (d)\ \vec{u}(x,0)=\vec{u}^{0}(x),\end{cases}$$

where,

$\vec{x}=(x_1\ ,\mathrm{x}_2,x_3)\in\Omega$: spatial coordinate in Ω;

$\vec{u}(x,t)$: the velocity field of the flow;

$\vec{u}_{\partial\Omega}(y,t)$: fluid velocity on the surface of a rotating obstacle;

$y:=(y_1,y_2)\in\partial\Omega$;

$p(x,t)$: the fluid pressure;

ω : angular velocity of the obstruction rotation, assumed constant;

ρ : fluid mass density, assumed constant;

μ : coefficient of viscosity, assumed constant.

3. Weak formulation

Our test functions are selected from the following set:

$$\Phi:=\{\vec{u}(x,t)\in H^2(\Omega)\ \mathrm{x}\ [0,T):\nabla.\vec{u}(x,t)=0,\vec{u}_{\partial\Omega}(y,t)=\vec{\omega}\Lambda\vec{y},\vec{y}\in\Omega\},$$
$$\text{with compact support}[\vec{u}(x,t)]\subset\Omega.$$

In particular, our spaces of interest are $L^2(\Omega),[0,T))$ and $L^2(\partial\Omega),[0,T))$. through the *trace theorem*, it can be shown that there exist a bijection defined by $\vec{u}\mapsto\gamma_0\vec{u}$, where $\vec{u}\in L^2(\Omega),[0,T))$ and $\gamma_0\vec{u}\in L^2(\partial\Omega),[0,T))$, (see theorem 9.4; on page 41 of [7].

4. The energy form of the Statement of the Problem

To derive the energy form of the problem, we take the scalar product of 1(a) with the velocity filed $u(x,t)$ and obtains the following

$$\rho\frac{1}{2}\frac{d}{dt}\|\vec{u}\|^2_{L^2(\Omega)}+\frac{1}{2}\mu\|\nabla\vec{u}\|^2_{L^2(\Omega)}=(f,\vec{u})_{L^2(\Omega)} \tag{2}$$

(See appendix A.1 below for the original derivation)

By the appendix below and introduction above the energy statement will reduce to

$$\rho \frac{1}{2}\frac{d}{dt}\|\vec{u}\|^2_{L^2(\Omega)} + \frac{1}{2}\mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} = \rho(f,\vec{u})_{L^2(\Omega)} \tag{3}$$

Simplifying (3), we obtain the following energy statement for the problem:

$$E'(t) + \mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} = \rho(f,\vec{u})_{L^2(\Omega)} \ , \tag{4}$$

as the energy identity for the problem. For

$c_\rho > 0,$ and since Ω is bounded, we can deduce the Poincare inequality:

$$\|\nabla\vec{u}(t)\|^2_{L^2(\Omega)} \geq c_\rho\|\vec{u}(t)\|^2_{L^2(\Omega)} \tag{5}$$

(See pp.248-249 of [2]).

In view of (5), we rewrite (4) as follows:

$$E'(t) + \mu c_\rho\|\vec{u}(t)\|^2_{L^2(\Omega)} \leq \rho(f,\vec{u})_{L^2(\Omega)} \tag{6}$$

That is, $E'(t) + \mu c_\rho E(t) \leq 2\rho(f,\vec{u})_{L^2(\Omega)}$ (6)

Re-writing the inequality (6) in terms of the kinetic energy for the flow, we obtain the following first order linear differential inequality:

$$E'(t) + \frac{\mu c_\rho}{\rho}E(t) \leq 2\rho(f,\vec{u})_{L^2(\Omega)} \tag{7}$$

The solution of (7) is given by,

$$E(t) \leq \exp\left(-\frac{\mu c_\rho t}{\rho}\right)\int_0^t \exp\left(\frac{\mu c_\rho t}{\rho}\right)(f(x,t),\vec{u}(x,t))_{L^2(\Omega)}\,dt + C_E \exp\left(-\frac{\mu c_\rho t}{\rho}\right); t\in[0,T) \tag{8}$$

Remarks 4.1

The inequality (8) implies that $E(0) \leq C_E, t = 0$

This, in turn, implies that,

$$\|\vec{u}^{\,0}(x)\| \leq C_E \tag{9}$$

Further, we have,

$$\|D(\vec{u})\|^2_{L^2(\Omega)} \geq \frac{1}{4}\|\nabla\vec{u}\|^2_{L^2(\Omega)} \geq \frac{1}{4}c_\rho\|\vec{u}\|^2_{L^2(\Omega)} \tag{10}$$

(See (12) of [6])

By (12) on page 9 of [5].

$$E'(t) \leq -\mu\|D(\vec{u})\|\beta(t)$$

Where, $\text{Max}\beta(t) = \frac{3}{2}$ (see the bottom of page 9 in [5])

Then

$$E'(t) \leq \frac{-3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)}$$

By (5) this implies that,

$$\frac{-3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} + \mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} \leq 2\rho(f,\vec{u})_{L^2(\Omega)} \tag{11}$$

By A.1

$$\frac{\mu}{2}\|\nabla\vec{u}\|^2_{L^2(\Omega)} = -\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)}\text{, which implies that,}$$

$$\mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} = -2\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)} \tag{12}$$

Using (12), we can re-write (11) and obtain,

$$\frac{-3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} - 2\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)} \leq 2\rho(f,\vec{u})_{L^2(\Omega)}$$

5. The Riesz's representation for the problem.

We re-write (13) as follows:

$$(\frac{-3}{8}\mu c_\rho(\vec{u},\vec{u})_{L^2(\Omega} - 2\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)} \leq 2\rho(f,\vec{u})_{L^2(\Omega)}$$

This implies that,

$$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} \leq 2\rho(f,\vec{u})_{L^2(\Omega)} \tag{14}$$

Remarks 6.1.

(a) Until now, through the application of Poincare inequality theorem, our aim has been to establish the boundedness of the right hand side of (14).

(b) It is not hard to show that the left hand side of (14) is a bounded sesquilinear form.
(c) Thus, by the Riesz's representation theorem [7], there exists a bounded linear operator A such that,

$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} = (A\vec{u},\vec{u})_{L^2(\Omega)}$, from which we conclude that:

$$\frac{-3}{8}\mu c_\rho I - 2\mu\Delta = A \tag{15}$$

6. The characterization of the operator $\frac{-3\mu}{8}c_\rho I - 2\mu\Delta$

Proposition 7.1. The operator $\frac{-3\mu}{8}c_\rho I - 2\mu\Delta$ is self-adjoint and positive on Θ

Proof. Let $\vec{u},\vec{v}\in\Theta$, Δ^* be adjoint of Δ and for $\vec{u}=\vec{v}$

$$(\Delta\vec{u},\vec{v})_{L^2(\Omega)} = \int_{\partial\Omega}\gamma_0\vec{v}.\text{n}.\nabla_s(\gamma_0\vec{u})ds - \frac{1}{2}\int_\Omega(\nabla\vec{u}.\nabla\vec{v})dx,$$

$$(\vec{u},\Delta^*\vec{v})_{L^2(\Omega)} = \int_{\partial\Omega}\gamma_0\vec{u}.\text{n}.\nabla_s^*(\gamma_0\vec{v})ds - \frac{1}{2}\int_\Omega(\nabla^*\vec{v}.\nabla\vec{u})dx,$$

Thus,

$(\Delta\vec{u},\vec{v})_{L^2(\Omega)} = (\vec{u},\Delta^*\vec{v})_{L^2(\Omega)}$, implies that,

$$\int_{\partial\Omega}\gamma_0\vec{v}.\text{n}.\nabla_s(\gamma_0\vec{u})ds - \frac{1}{2}\int_\Omega(\nabla\vec{u}.\nabla\vec{v})dx = \int_{\partial\Omega}\gamma_0\vec{u}.\text{n}.\nabla_s^*(\gamma_0\vec{v})ds - \frac{1}{2}\int_\Omega(\nabla^*\vec{v}.\nabla\vec{u})dx,$$

and since

$$\int_{\partial\Omega}\gamma_0\vec{v}.\text{n}.\nabla_s(\gamma_0\vec{u})ds = \int_{\partial\Omega}\gamma_0\vec{u}.\text{n}.\nabla_s^*(\gamma_0\vec{v})ds = 0, \text{due to } \gamma_0\vec{u}.\text{n} = 0.$$

$-\frac{1}{2}\int_\Omega(\nabla\vec{u}.\nabla\vec{v})dx = -\frac{1}{2}\int_\Omega(\nabla^*\vec{v}.\nabla\vec{u})dx,$ The latter equality implies that,

$\nabla\vec{v} = \nabla^*\vec{v}$, and the result follows.

Further, we consider $\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)}$:

$$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} = \frac{-3}{8}\mu c_\rho(\vec{u},\vec{u})_{L^2(\Omega)} - 2\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)}$$

$$= \frac{-3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} - 2\mu\left[\int_{\partial\Omega}\gamma_0\vec{u}.\text{n}.\nabla_s(\gamma_0\vec{u})ds - \frac{1}{2}\int_\Omega(\nabla\vec{u}.\nabla\vec{u})dx\right]$$

Thus,

$$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} = \frac{3}{8}\mu(-)c_\rho\|\vec{u}\|^2_{L^2(\Omega)} + \mu\int_\Omega(\nabla\vec{u}.\nabla\vec{u})dx - 2\mu\left[\int_{\partial\Omega}\gamma_0\vec{u}.\mathrm{n}.\nabla_s(\gamma_0\vec{u})ds\right]$$

$$= \mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} - \frac{3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} - 2\mu\left[\int_{\partial\Omega}\gamma_0\vec{u}.\mathrm{n}.\nabla_s(\gamma_0\vec{u})ds\right]$$

At the interaction between the fluid flow and rotating obstacle, we have $\gamma_0\vec{u}.n = 0$, thus

$$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} = \mu\|\nabla\vec{u}\|^2_{L^2(\Omega)} - \frac{3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} \geq \mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} - \frac{3}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)},$$

On applying the Poincare inequality.

Therefore,

$$\left((-\frac{3}{8}\mu c_\rho I - 2\mu\Delta)\vec{u},\vec{u}\right)_{L^2(\Omega)} \geq \frac{5}{8}\mu c_\rho\|\vec{u}\|^2_{L^2(\Omega)} \geq 0, \text{ the result follows.}$$

Proposition 7.2: $\frac{-3\mu}{8}c_\rho I - 2\mu\Delta$ is invertible and $(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)^{-1}$ is a bounded linear operator on Θ.

Proof:

$$\text{Let } \vec{u} \in Ker\,(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)\vec{u}. \text{ Then, } (\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)\vec{u} = 0,$$

This implies that

$$\frac{-3}{8}\mu c_\rho\vec{u} - 2\mu\Delta\vec{u} = 0 \Leftrightarrow \frac{-3}{8}\mu c_\rho\vec{u} = 2\mu\Delta\vec{u} = 0 \Leftrightarrow \vec{u} = 0. \text{ Then,}$$

$\ker(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta) = \{0\}$, and hence $(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)^{-1}$ existed.

We first prove the linearity of the inverse operator before we prove that it is bounded. It will be linear if:

$$1.(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2) = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_1 + (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_2$$

$$2..(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\lambda\vec{u}) = \lambda(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}$$

Now, to prove that (1): we put $.(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2) = \vec{w}, \vec{w} \in \Theta. then$
$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w} = \vec{u}_1 + \vec{u}_2$

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_1 = \vec{w}_1, \vec{w}_1 \in \Theta, then$$

$$(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}_1 = \vec{u}_1, and, (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_2 = \vec{w}_2, \vec{w}_2 \in \Theta, then$$

Put $(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}_2 = \vec{u}_2, therefore$

$$\vec{w}_1 + \vec{w}_2 = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_1 + (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}_2, and$$

$$\vec{u}_1 + \vec{u}_2 = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}_1 + (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}_2,$$

since $\frac{-3}{8}\mu c_\rho I - 2\mu\Delta$ is linear, then $\vec{w}_1 + \vec{w}_2 = .(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2)$,substituting it in place of $\vec{w}_1 + \vec{w}_2$ above, we have the results.

To prove (2): Put $(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u} = \vec{w}, \vec{w} \in \Theta, then$
$\vec{u} = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}$

Multiplying both sides by a costant λ, we have

$$\lambda\vec{u} = \lambda(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\vec{w}$$

$$\lambda\vec{u} = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)\lambda\vec{w}$$

$$\lambda\vec{w} = (\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}(\lambda\vec{u})$$

Replacing $\vec{w}$ by $(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}$ the results follows.

In view of the corollary on page 96 of [7], $(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta)^{-1}\vec{u}$,

Is bounded on $\boldsymbol{\Theta}$.

Proposition 7.3: $\frac{-3\mu}{8}c_\rho I - 2\mu\Delta$ is a compact operator on Θ .

Proof: since $\Omega \subset R^3$ and Θ is finite dimensional. By the corollary on page 407 of [7], $\frac{-3\mu}{8} c_\rho I - 2\mu\Delta$ is compact on Θ. By Reisz Representation Theorem [7], we conclude that A is also compact on Θ.

Proposition 7.4: The operator $(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta)^{-1} A$ is compact on Θ.

Proof: By Reisz Representation Theorem [7] and proposition 7.3, A is a compact operator on Θ. By proposition 2, $(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta)^{-1} : \Theta \to \Theta$ is a bounded linear operator. By theorem 8.3-2 in [7], then $(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta)^{-1} A$ is also compact and the results follows.

7. Existence and Uniqueness for the Solution to the Problem

Next, using (13), and in keeping with the requirements by the Leray-Schauder fixed-point theorem [8], we construct the following form for the problem:

$$(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} A\vec{u}, \text{ where } \lambda \in (0,1). \tag{16}$$

Our aim is to show that (16) has a unique Leray-Schauder fixed-point [8], which is the solution to the problem:

Firstly, we state and prove the following lemma:

Lemma 7.1

The solution to (13) is uniformly bounded in Θ.

Proof:

$$\vec{u} = \lambda(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)^{-1} A\vec{u}, \text{ where } \lambda \in (0,1), \text{implies that,}$$

$$(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta)\vec{u} = \lambda A\vec{u}, \text{which, in turn, implies that,}$$

$$\left\| \frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right\|_{W^2(\Omega)} \|\vec{u}\|_{W^2(\Omega)} \le |\lambda| \|A\vec{u}\|_{W^2(\Omega)} \le \|A\vec{u}\|_{W^2(\Omega)}, \text{since } |\lambda| < 1. \tag{17}$$

By the Rieszs representation theorem, A is also bounded, and hence, there exists $\eta > 0$, such that $\|A\vec{u}\|_{W^2(\Omega)} \le \eta$.

By (17), this implies that,

$$\|\vec{u}\|_{W^2(\Omega)} \le \frac{\eta}{\left\| \frac{-3}{8}\mu c_\rho I - 2\mu\Delta \right\|_{W^2(\Omega)}}, \text{ and the results follows.} \tag{18}$$

Main Theorem 7.2:
There exists a unique fixed-point for (13), which is the solution to (1).

Proof:
By lemma 8.1 and proposition 7.4, according to the leray-Schauder fixed-point theorem [8], (13) has a uniformly bounded solution.

To prove uniqueness:

Suppose (3) has two solutions $\vec{u}$ and $\vec{v}$.

Then,

$$\|\vec{u}-\vec{v}\|_{W^2(\Omega)} = \lambda \left\| \left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)^{-1} A(\vec{u}-\vec{v}) \right\|_{W^2(\Omega)}$$

$$\le |\lambda| \left\| \left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)^{-1} A \right\|_{W^2(\Omega)} \|\vec{u}-\vec{v}\|_{W^2(\Omega)},$$

due to boundednes

$$< \left\| \left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)^{-1} A \right\|_{W^2(\Omega)} \|\vec{u}-\vec{v}\|_{W^2(\Omega)};\ 0 < |\lambda| < 1$$

$$= \mathrm{I}\|\vec{u}-\vec{v}\|_{W^2(\Omega)};\ \text{which is impossible}.$$

Hence, $\vec{u} = \vec{v}$

In conclusion we say the initial velocity of the fluid is uniformly bounded. The fluid will rotate with the obstacle and decreases to zero as time goes to infinity**.**

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APPENDIX

A.1: Energy form of the problem is given by,

$$\rho(\frac{\partial\vec{u}}{\partial t},\vec{u})_{L^2(\Omega)}+\rho((\vec{u}.\nabla)\vec{u},\vec{u})_{L^2(\Omega)}=-(\nabla p,\vec{u})_{L^2(\Omega)}+\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)}+((\vec{\omega}\Lambda\vec{x}).\nabla\vec{u})\vec{u}+\rho(f,\vec{u})_{L^2(\Omega)},$$

Using integration by part and divergence theorem, we have,

$$(a)\ \rho(\frac{\partial\vec{u}}{\partial t},\vec{u})_{L^2(\Omega)}=\int_\Omega\rho(\frac{\partial\vec{u}}{\partial t},\vec{u})dx$$
$$=\rho\frac{d}{dt}\int_\Omega\vec{u}.\vec{u}$$
$$=\frac{\rho}{2}\frac{d}{dt}\|\vec{u}\|^2_{L^2(\Omega)}$$

$$(b)\ \rho((\vec{u}.\nabla)\vec{u},\vec{u})_{L^2(\Omega)}=\rho\int_\Omega[(\vec{u}.\nabla)\vec{u},\vec{u}]$$
$$=\rho\sum_{m=1}^{3}\sum_{k=1}^{3}\frac{1}{2}\int_\Omega\vec{u}_k\frac{\partial(\vec{u}_m\vec{u}_i)}{\partial x_k}dx_m$$
$$=\frac{\rho}{2}\left[\int_{\partial\Omega}(|\gamma_0\vec{u}.n||\gamma_0\vec{u}|^2)ds-\int_\Omega(|\vec{u}|^2\nabla.\vec{u})dx\right]$$

From the no-slip condition and 1(c) above, the equation collapse to zero.

$$(c)\ \mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)} = \mu\int_{\Omega}(\vec{u}\Delta\vec{u})dx$$

$$= \mu\int_{\partial\Omega}\gamma_0\vec{u}.n\,(\gamma_0\vec{u})ds - \frac{\mu}{2}\int_{\Omega}(\nabla\vec{u}.\nabla\vec{u})dx$$

Again from no-slip condition, the first part of the equation is zero, hence

$$\mu(\Delta\vec{u},\vec{u})_{L^2(\Omega)} = -\frac{\mu}{2}\int_{\Omega}(\nabla\vec{u}.\nabla\vec{u})dx$$

$$= -\frac{\mu}{2}\|\nabla\vec{u}\|^2_{L^2(\Omega)}$$

$$(d) - \nabla p,\vec{u} = \int_{\Omega}\nabla p.\vec{u}dx$$

$$= \int_{\partial\Omega}\mathrm{p}\gamma_0\vec{u}.nds + \int_{\Omega}p(\nabla.\vec{u})dx$$

From the no-slip condition and 1(c) above, the equation collapse to zero.

$$(e)\ ((\vec{\omega}\Lambda\vec{x}).\nabla\vec{u}),\vec{u} = (\gamma_0\vec{u}.\nabla\vec{u}),\vec{u}$$

$$= \left[\int_{\partial\Omega}(\gamma_0\vec{u}.n.\gamma_0\vec{u})ds - \int_{\Omega}\gamma_0\vec{u}(\nabla.\vec{u})dx\right],\vec{u}$$

From the no-slip condition and 1(c) above, the equation collapse to zero. Now the energy form of the problem becomes,

$$\frac{\rho}{2}\frac{d}{dt}\|\vec{u}\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|\nabla\vec{u}\|^2_{L^2(\Omega)} = \rho(f,\vec{u})_{L^2(\Omega)}$$