

On The 3D Incompressible Navier-stokes Flows Around a Rotating Obstacle

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Abstract

We consider the motion of an incompressible viscous fluid, governed by the well known Navier-Stokes system of equations, in an exterior domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega = \Gamma$, which is assumed to be infinitely differentiable. The open bounded domain itself is assumed to possess a cone property [1]. When the rotation of an obstacle is taken into account, for a 3D case only the existence of the weak solutions has been confirmed in [4]. In this paper, we proceed one step further and confirm the uniqueness of that weak solution, using the so-called “energy method”.

Keywords: viscous flow, rotating obstacle.

1. Introduction

The setting of the problem is 3D viscous, incompressible Laminar flow around a smooth impenetrable regular obstruction with a fixed axis of rotation slightly away from the centre of the channel of flow. The result of fluid flow with an obstacle is that, we end up with a net rotation (at angular velocity ω) in a particular direction. When the fluid flow is in contact with the rotating obstacle, the normal component of the flow surface velocity, $\gamma_0 \bar{u} \mathbf{n} = 0$. All the other surface velocity components are tangential to boundary of the rotating obstacle.

2. Statement of the Problem

We look for the unique solution $\bar{u}(x, t) \in H^2(\Omega) \times [0, T), T < \infty$, such that,

$$(1) \left\{ \begin{array}{l} (a) \rho \frac{\partial \bar{u}}{\partial t} + \rho(\nabla \cdot \bar{u})\bar{u} = -\nabla p + \mu\Delta\bar{u} + (\bar{\omega}\Lambda\bar{x}) \cdot \nabla\bar{u} + \rho f \\ \text{Subject to :} \\ (b) \bar{u}_{\partial\Omega} = \bar{\omega}\Lambda\bar{y}; \\ (c) \nabla \cdot \bar{u} = 0; \\ (d) \bar{u}(x,0) = \bar{u}^0(x), \end{array} \right.$$

where,

$\bar{x} = (x_1, x_2, x_3) \in \Omega$: spatial coordinate in Ω ;

$\bar{u}(x, t)$: the velocity field of the flow;

$\bar{u}_{\partial\Omega}(y, t)$: fluid velocity on the surface of a rotating obstacle;

$y := (y_1, y_2) \in \partial\Omega$;

$p(x, t)$: the fluid pressure;

ω : angular velocity of the obstruction rotation, assumed constant;

ρ : fluid mass density, assumed constant;

μ : coefficient of viscosity, assumed constant.

3. Weak formulation

Our test functions are selected from the following set:

$$\Phi := \{\bar{u}(x, t) \in H^2(\Omega) \times [0, T) : \nabla \cdot \bar{u}(x, t) = 0, \bar{u}_{\partial\Omega}(y, t) = \bar{\omega}\Lambda\bar{y}, \bar{y} \in \Omega\},$$

with compact support $[\bar{u}(x, t)] \subset \Omega$.

In particular, our spaces of interest are $L^2(\Omega), [0, T)$ and $L^2(\partial\Omega), [0, T)$. through the *trace theorem*, it can be shown that there exist a bijection defined by $\bar{u} \mapsto \gamma_0\bar{u}$, where $\bar{u} \in L^2(\Omega), [0, T)$ and $\gamma_0\bar{u} \in L^2(\partial\Omega), [0, T)$, (see theorem 9.4; on page 41 of [7].

4. The energy form of the Statement of the Problem

To derive the energy form of the problem, we take the scalar product of 1(a) with the velocity field $u(x, t)$ and obtains the following

$$\rho \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L^2(\Omega)}^2 + \frac{1}{2} \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = (f, \bar{u})_{L^2(\Omega)} \quad (2)$$

(See appendix A.1 below for the original derivation)

By the appendix below and introduction above the energy statement will reduce to

$$\rho \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L^2(\Omega)}^2 + \frac{1}{2} \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = \rho(f, \bar{u})_{L^2(\Omega)} \quad (3)$$

Simplifying (3), we obtain the following energy statement for the problem:

$$E'(t) + \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = \rho(f, \bar{u})_{L^2(\Omega)}, \quad (4)$$

as the energy identity for the problem. For

$c_\rho > 0$, and since Ω is bounded, we can deduce the Poincare inequality:

$$\|\nabla \bar{u}(t)\|_{L^2(\Omega)}^2 \geq c_\rho \|\bar{u}(t)\|_{L^2(\Omega)}^2 \quad (5)$$

(See pp.248-249 of [2]).

In view of (5), we rewrite (4) as follows:

$$E'(t) + \mu c_\rho \|\bar{u}(t)\|_{L^2(\Omega)}^2 \leq \rho(f, \bar{u})_{L^2(\Omega)} \quad (6)$$

$$\text{That is, } E'(t) + \mu c_\rho E(t) \leq 2\rho(f, \bar{u})_{L^2(\Omega)} \quad (6)$$

Re-writing the inequality (6) in terms of the kinetic energy for the flow, we obtain the following first order linear differential inequality:

$$E'(t) + \frac{\mu c_\rho}{\rho} E(t) \leq 2\rho(f, \bar{u})_{L^2(\Omega)} \quad (7)$$

The solution of (7) is given by,

$$E(t) \leq \exp\left(-\frac{\mu c_\rho t}{\rho}\right) \int_0^t \exp\left(\frac{\mu c_\rho t}{\rho}\right) (f(x, t), \bar{u}(x, t))_{L^2(\Omega)} dt + C_E \exp\left(-\frac{\mu c_\rho t}{\rho}\right); t \in [0, T] \quad (8)$$

Remarks 4.1

The inequality (8) implies that $E(0) \leq C_E, t = 0$

This, in turn, implies that,

$$\|\bar{u}^0(x)\| \leq C_E \quad (9)$$

Further, we have,

$$\|D(\bar{u})\|_{L^2(\Omega)}^2 \geq \frac{1}{4} \|\nabla \bar{u}\|_{L^2(\Omega)}^2 \geq \frac{1}{4} c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 \quad (10)$$

(See (12) of [6])

By (12) on page 9 of [5].

$$E'(t) \leq -\mu \|D(\bar{u})\| \beta(t)$$

Where, $\text{Max} \beta(t) = \frac{3}{2}$ (see the bottom of page 9 in [5])

Then

$$E'(t) \leq -\frac{3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2$$

By (5) this implies that,

$$\frac{-3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 + \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 \leq 2\rho(f, \bar{u})_{L^2(\Omega)} \quad (11)$$

By A.1

$$\frac{\mu}{2} \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = -\mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)}, \text{ which implies that,}$$

$$\mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = -2\mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)} \quad (12)$$

Using (12), we can re-write (11) and obtain,

$$\frac{-3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 - 2\mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)} \leq 2\rho(f, \bar{u})_{L^2(\Omega)}$$

5. The Riesz's representation for the problem.

We re-write (13) as follows:

$$\left(\frac{-3}{8} \mu c_\rho (\bar{u}, \bar{u})_{L^2(\Omega)} - 2\mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)} \right) \leq 2\rho(f, \bar{u})_{L^2(\Omega)}$$

This implies that,

$$\left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu \Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} \leq 2\rho(f, \bar{u})_{L^2(\Omega)} \quad (14)$$

Remarks 6.1.

- (a) Until now, through the application of Poincare inequality theorem, our aim has been to establish the boundedness of the right hand side of (14).

- (b) It is not hard to show that the left hand side of (14) is a bounded sesquilinear form.
- (c) Thus, by the Riesz's representation theorem [7], there exists a bounded linear operator A such that,

$$\left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} = (A\bar{u}, \bar{u})_{L^2(\Omega)}, \text{ from which we conclude that:}$$

$$\frac{-3}{8} \mu c_\rho I - 2\mu\Delta = A \quad (15)$$

6. The characterization of the operator $\frac{-3\mu}{8} c_\rho I - 2\mu\Delta$

Proposition 7.1. The operator $\frac{-3\mu}{8} c_\rho I - 2\mu\Delta$ is self-adjoint and positive on Θ

Proof. Let $\bar{u}, \bar{v} \in \Theta$, Δ^* be adjoint of Δ and for $\bar{u} = \bar{v}$

$$(\Delta\bar{u}, \bar{v})_{L^2(\Omega)} = \int_{\partial\Omega} \gamma_0 \bar{v} \cdot \mathbf{n} \cdot \nabla_s (\gamma_0 \bar{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla\bar{u} \cdot \nabla\bar{v}) dx,$$

$$(\bar{u}, \Delta^*\bar{v})_{L^2(\Omega)} = \int_{\partial\Omega} \gamma_0 \bar{u} \cdot \mathbf{n} \cdot \nabla_s^* (\gamma_0 \bar{v}) ds - \frac{1}{2} \int_{\Omega} (\nabla^*\bar{v} \cdot \nabla\bar{u}) dx,$$

Thus,

$$(\Delta\bar{u}, \bar{v})_{L^2(\Omega)} = (\bar{u}, \Delta^*\bar{v})_{L^2(\Omega)}, \text{ implies that,}$$

$$\int_{\partial\Omega} \gamma_0 \bar{v} \cdot \mathbf{n} \cdot \nabla_s (\gamma_0 \bar{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla\bar{u} \cdot \nabla\bar{v}) dx = \int_{\partial\Omega} \gamma_0 \bar{u} \cdot \mathbf{n} \cdot \nabla_s^* (\gamma_0 \bar{v}) ds - \frac{1}{2} \int_{\Omega} (\nabla^*\bar{v} \cdot \nabla\bar{u}) dx,$$

and since

$$\int_{\partial\Omega} \gamma_0 \bar{v} \cdot \mathbf{n} \cdot \nabla_s (\gamma_0 \bar{u}) ds = \int_{\partial\Omega} \gamma_0 \bar{u} \cdot \mathbf{n} \cdot \nabla_s^* (\gamma_0 \bar{v}) ds = 0, \text{ due to } \gamma_0 \bar{u} \cdot \mathbf{n} = 0.$$

$$-\frac{1}{2} \int_{\Omega} (\nabla\bar{u} \cdot \nabla\bar{v}) dx = -\frac{1}{2} \int_{\Omega} (\nabla^*\bar{v} \cdot \nabla\bar{u}) dx, \text{ The latter equality implies that,}$$

$\nabla\bar{v} = \nabla^*\bar{v}$, and the result follows.

Further, we consider $\left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)}$:

$$\begin{aligned} \left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} &= \frac{-3}{8} \mu c_\rho (\bar{u}, \bar{u})_{L^2(\Omega)} - 2\mu (\Delta\bar{u}, \bar{u})_{L^2(\Omega)} \\ &= \frac{-3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 - 2\mu \left[\int_{\partial\Omega} \gamma_0 \bar{u} \cdot \mathbf{n} \cdot \nabla_s (\gamma_0 \bar{u}) ds - \frac{1}{2} \int_{\Omega} (\nabla\bar{u} \cdot \nabla\bar{u}) dx \right] \end{aligned}$$

Thus,

$$\begin{aligned} \left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} &= \frac{3}{8} \mu (-) c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 + \mu \int_{\Omega} (\nabla \bar{u} \cdot \nabla \bar{u}) dx - 2\mu \left[\int_{\partial\Omega} \gamma_0 \bar{u} \cdot n \cdot \nabla_s (\gamma_0 \bar{u}) ds \right] \\ &= \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 - 2\mu \left[\int_{\partial\Omega} \gamma_0 \bar{u} \cdot n \cdot \nabla_s (\gamma_0 \bar{u}) ds \right] \end{aligned}$$

At the interaction between the fluid flow and rotating obstacle, we have $\gamma_0 \bar{u} \cdot n = 0$, thus

$$\left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} = \mu \|\nabla \bar{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 \geq \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 - \frac{3}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2,$$

On applying the Poincare inequality.

Therefore,

$$\left(\left(-\frac{3}{8} \mu c_\rho I - 2\mu\Delta \right) \bar{u}, \bar{u} \right)_{L^2(\Omega)} \geq \frac{5}{8} \mu c_\rho \|\bar{u}\|_{L^2(\Omega)}^2 \geq 0, \text{ the result follows.}$$

Proposition 7.2: $\frac{-3\mu}{8} c_\rho I - 2\mu\Delta$ is invertible and $\left(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta \right)^{-1}$ is a bounded linear operator on Θ .

Proof:

$$\text{Let } \bar{u} \in \text{Ker} \left(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta \right) \bar{u}. \text{ Then, } \left(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta \right) \bar{u} = 0,$$

This implies that

$$\frac{-3}{8} \mu c_\rho \bar{u} - 2\mu\Delta \bar{u} = 0 \Leftrightarrow \frac{-3}{8} \mu c_\rho \bar{u} = 2\mu\Delta \bar{u} = 0 \Leftrightarrow \bar{u} = 0. \text{ Then,}$$

$$\text{ker} \left(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta \right) = \{0\}, \text{ and hence } \left(\frac{-3\mu}{8} c_\rho I - 2\mu\Delta \right)^{-1} \text{ existed.}$$

We first prove the linearity of the inverse operator before we prove that it is bounded. It will be linear if:

1. $\left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} (\bar{u}_1 + \bar{u}_2) = \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} \bar{u}_1 + \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} \bar{u}_2$
2. $\left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} (\lambda \bar{u}) = \lambda \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} \bar{u}$

$$.(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2) = \vec{w}, \vec{w} \in \Theta, \text{ then}$$

Now, to prove that (1): we put

$$(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w} = \vec{u}_1 + \vec{u}_2$$

$$(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}_1 = \vec{w}_1, \vec{w}_1 \in \Theta, \text{ then}$$

$$(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}_1 = \vec{u}_1, \text{ and, } (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}_2 = \vec{w}_2, \vec{w}_2 \in \Theta, \text{ then}$$

$$\text{Put } (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}_2 = \vec{u}_2, \text{ therefore}$$

$$\vec{w}_1 + \vec{w}_2 = (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}_1 + (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}_2, \text{ and}$$

$$\vec{u}_1 + \vec{u}_2 = (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}_1 + (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}_2,$$

since

$$\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta \text{ is linear, then } \vec{w}_1 + \vec{w}_2 = .(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}(\vec{u}_1 + \vec{u}_2)$$

,substituting it in place of $\vec{w}_1 + \vec{w}_2$ above, we have the results.

$$(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u} = \vec{w}, \vec{w} \in \Theta, \text{ then}$$

To prove (2): Put

$$\vec{u} = (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}$$

Multiplying both sides by a constant

λ , we have

$$\lambda\vec{u} = \lambda(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\vec{w}$$

$$\lambda\vec{u} = (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)\lambda\vec{w}$$

$$\lambda\vec{w} = (\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}(\lambda\vec{u})$$

Replacing \vec{w} by $(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}$ the results follows.

In view of the corollary on page 96 of [7], $(\frac{-3}{8}\mu c_{\rho}I - 2\mu\Delta)^{-1}\vec{u}$,

Is bounded on Θ .

Proposition 7.3: $\frac{-3\mu}{8}c_{\rho}I - 2\mu\Delta$ is a compact operator on Θ .

Proof: since $\Omega \subset R^3$ and Θ is finite dimensional. By the corollary on page 407 of [7], $\frac{-3\mu}{8}c_\rho I - 2\mu\Delta$ is compact on Θ . By Reisz Representation Theorem [7], we conclude that A is also compact on Θ .

Proposition 7.4: The operator $(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)^{-1}A$ is compact on Θ .

Proof: By Reisz Representation Theorem [7] and proposition 7.3, A is a compact operator on Θ . By proposition 2, $(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)^{-1} : \Theta \rightarrow \Theta$ is a bounded linear operator. By theorem 8.3-2 in [7], then $(\frac{-3\mu}{8}c_\rho I - 2\mu\Delta)^{-1}A$ is also compact and the results follows.

7. Existence and Uniqueness for the Solution to the Problem

Next, using (13), and in keeping with the requirements by the Leray-Schauder fixed-point theorem [8], we construct the following form for the problem:

$$\left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)^{-1}A\bar{u}, \text{ where } \lambda \in (0,1). \quad (16)$$

Our aim is to show that (16) has a unique Leray-Schauder fixed-point [8], which is the solution to the problem:

Firstly, we state and prove the following lemma:

Lemma 7.1

The solution to (13) is uniformly bounded in Θ .

Proof:

$$\begin{aligned} \bar{u} &= \lambda \left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)^{-1}A\bar{u}, \text{ where } \lambda \in (0,1), \text{ implies that,} \\ \left(\frac{-3}{8}\mu c_\rho I - 2\mu\Delta\right)\bar{u} &= \lambda A\bar{u}, \text{ which, in turn, implies that,} \\ \left\| \frac{-3}{8}\mu c_\rho I - 2\mu\Delta \right\|_{W^2(\Omega)} \|\bar{u}\|_{W^2(\Omega)} &\leq |\lambda| \|A\bar{u}\|_{W^2(\Omega)} \leq \|A\bar{u}\|_{W^2(\Omega)}, \text{ since } |\lambda| < 1. \end{aligned} \quad (17)$$

By the Riesz representation theorem, A is also bounded, and hence, there exists $\eta > 0$, such that $\|A\bar{u}\|_{W^2(\Omega)} \leq \eta$.

By (17), this implies that,

$$\|\bar{u}\|_{W^2(\Omega)} \leq \frac{\eta}{\left\| \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right) \right\|_{W^2(\Omega)}}, \text{ and the results follows.} \quad (18)$$

Main Theorem 7.2:

There exists a unique fixed-point for (13), which is the solution to (1).

Proof:

By lemma 8.1 and proposition 7.4, according to the lery-Schauder fixed-point theorem [8], (13) has a uniformly bounded solution.

To prove uniqueness:

Suppose (3) has two solutions \bar{u} and \bar{v} .

Then,

$$\begin{aligned} \|\bar{u} - \bar{v}\|_{W^2(\Omega)} &= \lambda \left\| \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} A(\bar{u} - \bar{v}) \right\|_{W^2(\Omega)} \\ &\leq |\lambda| \left\| \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} A \right\|_{W^2(\Omega)} \|\bar{u} - \bar{v}\|_{W^2(\Omega)}, \end{aligned}$$

due to boundednes

$$\begin{aligned} &< \left\| \left(\frac{-3}{8} \mu c_\rho I - 2\mu\Delta \right)^{-1} A \right\|_{W^2(\Omega)} \|\bar{u} - \bar{v}\|_{W^2(\Omega)}; \quad 0 < |\lambda| < 1 \\ &= \lambda \|\bar{u} - \bar{v}\|_{W^2(\Omega)}; \text{ which is impossible.} \end{aligned}$$

Hence, $\bar{u} = \bar{v}$

In conclusion we say the initial velocity of the fluid is uniformly bounded. The fluid will rotate with the obstacle and decreases to zero as time goes to infinity.

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APPENDIX

A.1: Energy form of the problem is given by,

$$\rho \left(\frac{\partial \vec{u}}{\partial t}, \vec{u} \right)_{L^2(\Omega)} + \rho \left((\vec{u} \cdot \nabla) \vec{u}, \vec{u} \right)_{L^2(\Omega)} = -(\nabla p, \vec{u})_{L^2(\Omega)} + \mu (\Delta \vec{u}, \vec{u})_{L^2(\Omega)} + ((\vec{\omega} \wedge \vec{x}) \cdot \nabla \vec{u}) \vec{u} + \rho (f, \vec{u})_{L^2(\Omega)},$$

Using integration by part and divergence theorem, we have,

$$\begin{aligned} (a) \quad \rho \left(\frac{\partial \vec{u}}{\partial t}, \vec{u} \right)_{L^2(\Omega)} &= \int_{\Omega} \rho \left(\frac{\partial \vec{u}}{\partial t}, \vec{u} \right) dx \\ &= \rho \frac{d}{dt} \int_{\Omega} \vec{u} \cdot \vec{u} \\ &= \frac{\rho}{2} \frac{d}{dt} \|\vec{u}\|_{L^2(\Omega)}^2 \end{aligned}$$

$$\begin{aligned} (b) \quad \rho \left((\vec{u} \cdot \nabla) \vec{u}, \vec{u} \right)_{L^2(\Omega)} &= \rho \int_{\Omega} [(\vec{u} \cdot \nabla) \vec{u}, \vec{u}] \\ &= \rho \sum_{m=1}^3 \sum_{k=1}^3 \frac{1}{2} \int_{\Omega} \vec{u}_k \frac{\partial (\vec{u}_m \vec{u}_i)}{\partial x_k} dx_m \\ &= \frac{\rho}{2} \left[\int_{\partial \Omega} (|\gamma_0 \vec{u} \cdot \vec{n}| |\gamma_0 \vec{u}|^2) ds - \int_{\Omega} (|\vec{u}|^2 \nabla \cdot \vec{u}) dx \right] \end{aligned}$$

From the no-slip condition and 1(c) above, the equation collapse to zero.

$$\begin{aligned}
 (c) \mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)} &= \mu \int_{\Omega} (\bar{u} \Delta \bar{u}) dx \\
 &= \mu \int_{\partial\Omega} \gamma_0 \bar{u} \cdot n (\gamma_0 \bar{u}) ds - \frac{\mu}{2} \int_{\Omega} (\nabla \bar{u} \cdot \nabla \bar{u}) dx
 \end{aligned}$$

Again from no-slip condition, the first part of the equation is zero, hence

$$\begin{aligned}
 \mu(\Delta \bar{u}, \bar{u})_{L^2(\Omega)} &= -\frac{\mu}{2} \int_{\Omega} (\nabla \bar{u} \cdot \nabla \bar{u}) dx \\
 &= -\frac{\mu}{2} \|\nabla \bar{u}\|_{L^2(\Omega)}^2
 \end{aligned}$$

$$\begin{aligned}
 (d) -\nabla p, \bar{u} &= \int_{\Omega} \nabla p \cdot \bar{u} dx \\
 &= \int_{\partial\Omega} p \gamma_0 \bar{u} \cdot n ds + \int_{\Omega} p (\nabla \cdot \bar{u}) dx
 \end{aligned}$$

From the no-slip condition and 1(c) above, the equation collapse to zero.

$$\begin{aligned}
 (e) ((\bar{\omega} \Lambda \bar{x}) \cdot \nabla \bar{u}), \bar{u} &= (\gamma_0 \bar{u} \cdot \nabla \bar{u}), \bar{u} \\
 &= \left[\int_{\partial\Omega} (\gamma_0 \bar{u} \cdot n \cdot \gamma_0 \bar{u}) ds - \int_{\Omega} \gamma_0 \bar{u} (\nabla \cdot \bar{u}) dx \right], \bar{u}
 \end{aligned}$$

From the no-slip condition and 1(c) above, the equation collapse to zero. Now the energy form of the problem becomes,

$$\frac{\rho}{2} \frac{d}{dt} \|\bar{u}\|_{L^2(\Omega)}^2 + \frac{\mu}{2} \|\nabla \bar{u}\|_{L^2(\Omega)}^2 = \rho(f, \bar{u})_{L^2(\Omega)}$$

