QMI 1500 QUANTITATIVE METHODS Chapter One

NUMBERS, VARIABLES AND OPERATIONS ON THESE NUMBERS

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Types of Numbers

Numbers, like fruits, are grouped into types. For Example, in fruit grouping we have apples. In this category we also have a variety of apples – namely, Golden Delicious, Top Red, Granny Smiths and/or Lady in Pink. However different, they are all still apples.

The analogous grouping in numbers would be that we have Real Numbers. These are all the numbers on a number line. Graphically, these are illustrated on Figure 1A.

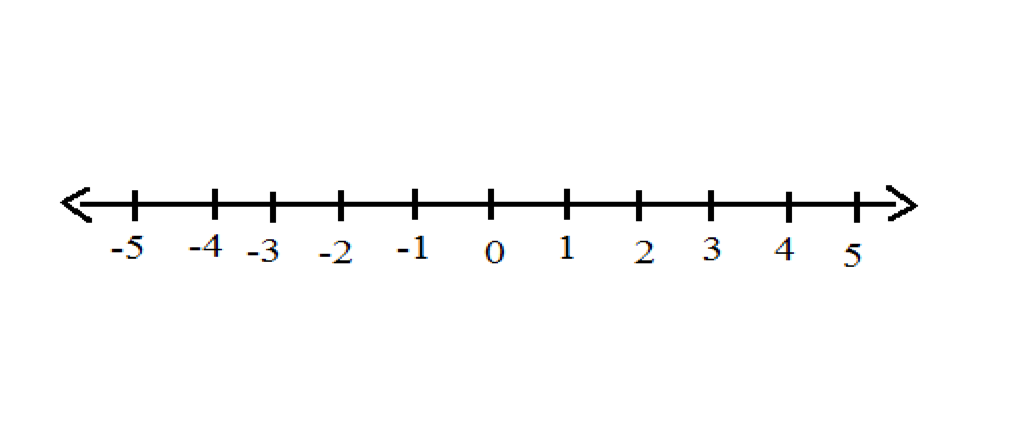

Figure 1A: Number Line

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These will include all the numbers in between as well, that is, the fractions. There are also subsets of the real numbers such as the integers.

The integers are simply the numbers that appear on the number line (those we can see) in Figure 1A,

$$\{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}.$$

Of the integer set going to the far left we see the negative integers, then to the right we have the non-negative integers (Counting Numbers) and positive integers (Natural Numbers), respectively, they are

$$\{\dots, -5, -4, -3, -2, -1\},$$

$\{0, 1, 2, 3, 4, 5, 6, \dots\}$ and lastly

$$\{1, 2, 3, 4, 5, 6, 7, \dots\}.$$

On the number line between each pair of integers and as part of the real numbers we have other numbers, uncountable numbers. These uncountable numbers together with the integers are known as the real numbers.

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All numbers on the number line including all those between the integers are collectively known as the real numbers. As an example, consider Figure 1B.

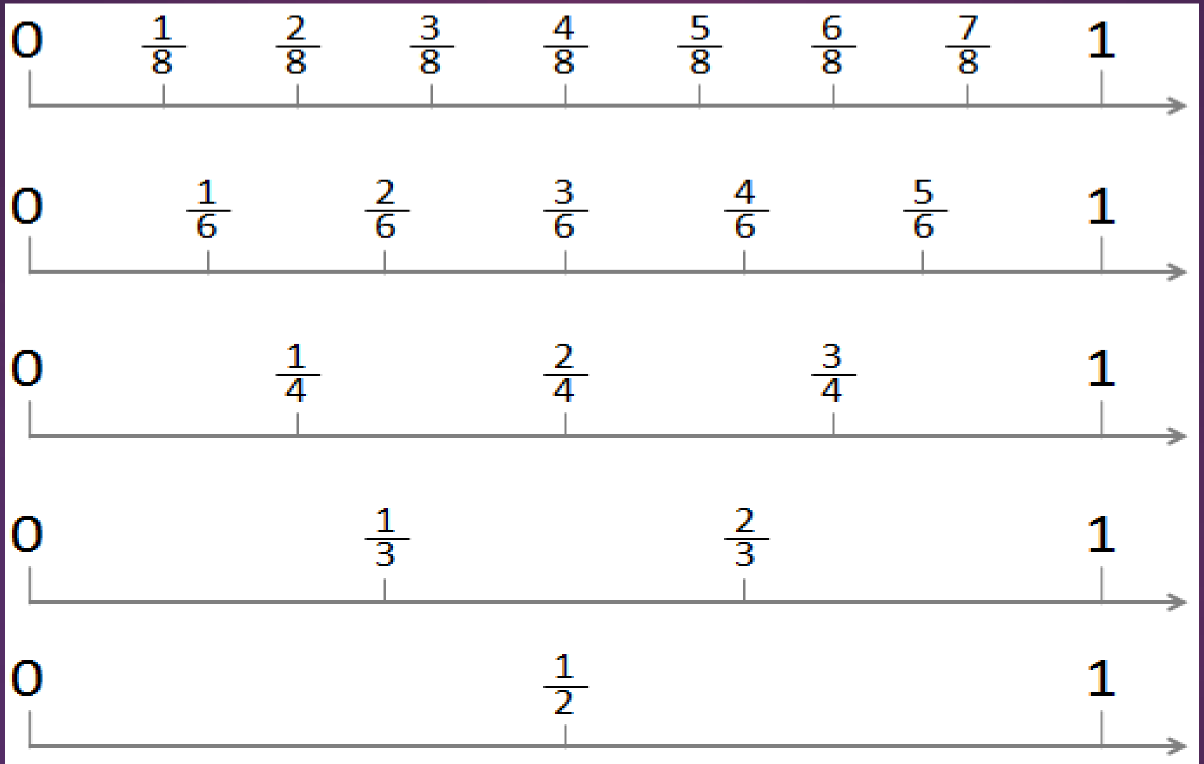

Figure 1B: Fractions between Integers

Next we consider rounding off real numbers.

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Fractions, Rounding and Variables

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Let's begin by defining a fraction. A fraction is a number with a value between two integers. Examples include,

$$\{-12\frac{8}{10}; -\frac{1}{100}; \frac{1}{2}; \frac{3}{2}; 5\frac{1}{2}; 100\frac{2}{3}; \ldots\}.$$

There are different types of fractions. First we have a proper fraction. This is one with the numerator (the top number) smaller than the bottom number (denominator) like these

$$\{\pm\frac{5}{100}; \pm\frac{10}{18}; \frac{\pm 45}{50}\}.$$

Secondly, suppose the numerator was greater than the denominator (ignoring the sign of the fraction) as in these numbers

$$\{\frac{85}{13}; -\frac{9}{3}; \pm\frac{5}{2}\},$$

our fractions are then known as improper fractions. If there is a whole number (an integer to be precise) to the left of our fraction then we have a mixed fraction or mixed number. These are of the form,

$\{\pm 5\frac{1}{2};\ \pm 8\frac{3}{20}\}$.

When there is a negative sign before the mixed number then think of the value as,

$$-5\frac{7}{14} = -\left(5\frac{7}{14}\right) = -\frac{5\times 14+7}{14} = -\frac{77}{14} = -\frac{11}{2} \ldots .\ (1)$$

The green shows how to convert a mixed number to an improper fraction. It is the whole number multiplied by the denominator added to the numerator over the denominator.

The pink is the reduced (simplified) fraction to one of numbers of lesser value.

Lastly, there is what we call a decimal fraction. These are fractions in comma format. For instance, these are:

$$1{,}45;\ 45{,}23;\ 3{,}985475.$$

To work with fewer numbers we perform what is known as rounding off. To round of to any number of decimal places, we count (starting from the first number after the decimal) the number of numbers we want after the decimal sign, (,). We then look at the number directly or immediately after – if the number is 5, 6, 7, 8, 9 we increase the number that comes before it by one and remove the rest of the digits to the right of that number.

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To illustrate, round off

$$15{,}354876$$

to four decimal places. The four decimal places will then be

$$15{,}\underline{3548}76.$$

The number immediately after is the Seven. So we increase the one before the Seven, which is the Eight by one to 9 and discard everything that comes thereafter. This summarises to

$$15{,}3549$$

Our 8 became a 9.

A variable is a symbol, normally a letter e.g. $a, b, c, d, e, \ldots$ that represent an indefinite or unknown real number and value. Variables represent numbers so they are essentially numbers, and work as numbers do. We can perform manipulations on numbers and variables alike using operations off addition, +, subtraction, -, multiplication, ×, division, ÷, exponents, x^n and roots, $\sqrt[n]{x}$.

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OPERATIONS ON NUMBERS AND VARIABLES

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Operations on Numbers and Variables.

Beginning with a demonstration of the operations of addition and subtraction,

$$8 + 22 = 22 + 8 = 30,$$

This exemplifies the Commutative Property of addition $(a + b = b + a)$.

$$17 – 4 \neq 4 – 17 \text{ (Not Equal.)}$$

The order in which we add is not important, however, the order in which we subtract is relevant as shown above. To introduce variables,

$$x + 13,$$

$$20 - z.$$

Together, the operations of {+, -} are

$$y + 13 - x + 12 - 5.$$

There is an order of execution among all operations but between the two of addition and subtraction we execute from left to right. In other words, take

$$y + 13 - x + 12 - 5.$$

In addition suppose $y = 3$ and $x = 12$. If we substitute for the real values we get

$$-3 + 13 - 12 + 12 - 5 = 10 - 12 + 12 - 5$$
$$= -2 + 12 - 5$$
$$= 10 - 5$$
$$= 5.$$

When we have variables of different kinds in one expression such as

$$z - y + x.$$

We cannot add or subtract these. For example, imagine z is pineapples, y is apples and x is peaches. Now

$$z - y$$

will be what is left if we remove apples, y, from pineapples, z. It makes no sense. If the variables are the same, however, then we can add and subtract like

$$2y - y + 3x + 7x = (2 - 1)y + (3 + 7)x = y + 10x$$

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Addition comes with a property which we call the Associative Law of addition, $(a + (b + c) = (a + b) + c$. This means that if we have

$$(7 + 3) + 12 = 7 + (3 + 12).$$

It is irrelevant which two you add first. This is also the case with subtraction. Introducing the Associative Law of Subtraction,

$$(x - y) - b = x + (-y - b).$$

It is interesting how we add x and $-y - b$. This will be clearer once we introduce multiplication and division.

Example 1A

What is the difference of 8 and 5?

That is,

$$8 - 5 = 3.$$

Example 1B

Work out the difference between – 8 and – 5?

$$-8 - (-5) = -8 + 5 = -3.$$

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Multiplication and division have a few rules of their own. Starting with what happens to the plus, +, and minus, -, when we multiply or divide.

The product (multiplication) of two numbers with opposite signs (+ and –) will produce a negative value.

Example 1C

$$-6 \times (+2) = +2 \times (-6) = -12$$

By the Commutative Law of multiplication ($a \times b = b \times a$) and

$$+6 \times (-12) = -12 \times 6 = -72.$$

If the signs are the same {+, +} or {-, -} the result of a multiplication should be positive.

Example 1D

$$(-4) \times (-5) = (-5) \times (-4) = +(4 \times 5) = 20$$

Furthermore,

$$(+13) \times (+3) = +(13 \times 3) = 39.$$

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For opposite signs, plus and minus, the division of two numbers with opposite signs produces a negative value.

Example 1E

The number being divided is called the dividend, that dividing is the divisor and the answer is the quotient.

$$-45 \div (+3) = -15.$$

Similarly,

$$+8 \div (-8) = 64.$$

Dividend × Divisor = Quotient!

For the quotient of numbers or variables with the same signs the value will be positive.

Example 1F

$$81 \div 9 = +9$$

and

$$-100 \div (-5) = +20.$$

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The Commutative and Associative Laws apply for multiplication but not for division.

For products,

$$(-6 \times 10) \times (-2) = -6 \times \big(10 \times (-2)\big) = -6 \times (-20) = 120.$$

By the Associative Law of multiplication $(a \times b) \times c = a \times (b \times c)$. Division does not, unfortunately, adhere to the Commutative and Associative Laws. Because,

Example 1G

$$9 \div 3 \neq 3 \div 9 \text{ (Not Commutative)}$$

Moreover,

$$9 \div (3 \div 3) \neq (9 \div 3) \div 3$$
$$9 \div 1 \neq 3 \div 3$$
$$9 \neq 1. \text{ (Not Associative)}$$

Again these operations apply for both numbers and variables alike.

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Consider the expression,

$$5y - 7.$$

Here, we have an expression arranged as

Coefficient × Variable – Constant.

Powers or Exponents are a way of multiplying the same number as many times as you would like. For instance

$$2^3 = 2 \times 2 \times 2.$$

The point is then, for a number or variable (x) with a power, n, such that n is an integer.

$$x^n = x \times x \times x \times \cdots \times x \ (n \text{ times.})$$

So

$$5^2 = 5 \times 5.$$

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It is displayed in the following way,

$$\text{Base}^{\text{Exponent}}.$$

These numbers can be any of the real numbers but more importantly the exponent can be any integer. Assume our power was negative for a variable, v.

$$v^{-3} = \frac{1}{v^3}$$

In essence,

$$v^{-n} = \frac{1}{v^n}.$$

Exponents have rules (Laws of Exponents) on how to work them out. The first two work only if the base is the same.

Multiplication ($a^n \times a^m = a^{n+m}$)

Division ($a^n \div a^m = a^{n-m}$)

The rest can be summarized as

$$(a^n)^m = a^{n\times m},$$

$$(ab)^n = a^n \times b^n.$$

When one has a minus or plus between numbers that are powered then you cannot simplify it any further.

$$(a \pm b)^n = (a \pm b)^n.$$

Roots are about finding which number multiplied by itself as many times as we need will give us the number under the root sign. Exemplar,

$$\sqrt[2]{4} = \sqrt{2\times 2} = \sqrt{2^2} = 2.$$

This tells us that the number we need to power by two to get four is two. So

$$\sqrt[n]{x}\ldots\text{(Expression 1A)}$$

means what number can we multiply by itself n times to get a total of x. We can also interpret roots as powers.

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In Expression 1A,

$$\sqrt[n]{x} = x^{\frac{1}{n}}.$$

Your calculator can work these fractional powers out but to do it by hand the principles remain the same. n in this case is a natural number.

Example 1H

Using these different ways of working-out roots, one by manually finding the number and the other by using powers we conclude that,

$$\sqrt[3]{27} = \sqrt[3]{3 \times 3 \times 3} = 3$$

$$\sqrt[3]{8} = (2^3)^{\frac{1}{3}} = 2^{3 \times \frac{1}{3}} = 2^1 = 2$$

remembering that

$$(a^n)^m = a^{n \times m}.$$

I understand that many of you do not have the recommended financial calculator. This brings me to the Logarithms. What is important for you to know is the following Log Property:

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if we had

$$8^{rt}$$

and we wanted to drop r or t we'd $\log$ the term and use the property that, for any

$$a^x$$

$$\log(a^x) = x[log\ (a)].$$

This will become more pragmatic once we consider equations and formulae.

To reiterate, all these operations on numbers and variables have an order of execution. By priority, the execution sits as follows

High Priority → Low Priority

Brackets Of Division Multiplication Addition Subtraction

Of includes Percentages, Fraction and Powers (Exponents)

Diagram 2A: BODMAS

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Example 1I

$$(50 - (3 + 4 \times 6) + 15 \div 5 \times 3) - (36 \div 4 + 2)$$
$$= (50 - 27 + 15 \div 5 \times 3) - 11$$
$$= (50 - 27 + 3 \times 3) - 11$$
$$= (50 - 27 + 9) - 11$$
$$= 32 - 11$$
$$= 21$$

Example 1J

$$(13 + 12) \times (20 + (8 \times 5 - 10) \div 5)$$
$$= 25 \times (20 + (40 - 10) \div 5)$$
$$= 25 \times (20 + 30 \div 5)$$
$$= 25 \times (20 + 6)$$
$$= 25 \times 26$$
$$= 650$$

Example 1K

We can now introduce other operations into our expressions.

$$\sqrt[3]{4\frac{12}{125}} + \sqrt[2]{3 \times \sqrt{(4^2 + 3^2)} + 7^3} - \frac{3}{5}$$

$$= \sqrt[3]{\frac{4 \times 125 + 12}{125}} + \sqrt{3 \times \sqrt{25} + 49} - \frac{3}{5}$$

$$= \sqrt[3]{\frac{512}{125}} + \sqrt{3 \times 5 + 49} - \frac{3}{5}$$

$$= \sqrt[3]{\frac{8^3}{5^3}} + \sqrt{15 + 49} - \frac{3}{5}$$

$$= \frac{\sqrt[3]{8^3}}{\sqrt[3]{5^3}} + \sqrt{8^2} - \frac{3}{5}$$

$$= \frac{8}{5} + 8 - \frac{3}{5}$$

$$= 8 + \frac{8}{5} - \frac{3}{5}$$

$$= 8 + \frac{5}{5}$$

$$= 8 + 1$$

$$= 9$$

MORE OPERATIONS ON NUMBERS

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Factors, Multiples, Fractions, Decimals, Percentages and Ratios

A factor of a number is a number by which it is divisible. This means by which it can be divided leaving no remainder.

Example 1L

Factors of 6 are $\{-6, -3, -2, -1, 1, 2, 3, 6\}$. Because 6 can be divided by each of the numbers in that set.

Example 1M

The factors of 50 are $\{-50, -25, -10, -5, -2, -1, 1, 2, 5, 10, 25, 50\}$.

Example 1N

For numbers such as 13 the factors are $\{-13, -1, 1, 13\}$.

This is basically only divisible by 1 and itself, 13. Numbers of this kind are called prime numbers. These are numbers divisible by 1 and themselves only.

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The LCM (lowest common multiple) of 2 or n numbers is the smallest number that is divisible by all those 2 or n numbers. Before finding the LCM we need one more concept named "prime factorization."

The prime factorization of a number is the product of it's prime factors whose value is the number itself.

Example 10

The prime factorize of $5, 12, 30, 70$ and 100.

$5 = 5$ (Five is already Prime)

$$12 = 2 \times 2 \times 3$$
$$30 = 2 \times 3 \times 5$$
$$70 = 2 \times 5 \times 7$$
$$100 = 2 \times 2 \times 5 \times 5$$

Finding the LCM we employ prime factorization in the following manner. Assume we want the LCM of 12 and 30. We start by writing these in prime factor form like so,

$$12 = 2 \times 2 \times 3,$$

$$30 = 2 \times 3 \times 5.$$

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We then multiply the greatest products of each prime number that appear in the prime factorizations.

The greatest product of two here is from the twelve and it is

$$2 \times 2.$$

That of three is just 3 because they are equal in both 12 and 30. Finally we only have 5 left and the highest product of 5 is from 30 and it's 5 itself.

The

$$LCM(12, 30) = (2 \times 2) \times 3 \times 5$$
$$= 60.$$

The GCF (greatest common factor) of two numbers is the highest number that can divide the two numbers. For the GCF of 12 and 30 we again utilize the prime factorization idea. We know that

$$12 = 2 \times 2 \times 3,$$

$$30 = 2 \times 3 \times 5.$$

The GCF is then the product of the smallest product of each prime number.

The least product of two is from 30 and it is 2. The lowest product of three is simply 3, since they are the same in 12 and 30. Then the smallest product of 5 is 5^0 from 12 because there are no fives.

The

$$GCF(12,30) = 2 \times 3 \times 5^0$$
$$= 6 \times 1$$
$$= 6.$$

The LCM and GCF are important for working with fractions, specifically to add and subtract fractions and to reduce or simplify expressions of fractions, respectively.

If we would like to add or subtract two fractions we can only hope to achieve that if their denominators are the same and if this is not the case we then have to convert the fractions so as to get the denominators to be the same.

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Example 1P

$$\left(5\frac{1}{4}+2\frac{1}{5}\right)-\left(2\frac{1}{3}+3\frac{1}{6}\right)=\left(\frac{21}{4}+\frac{11}{5}\right)-\left(\frac{7}{3}+\frac{19}{6}\right)$$

The LCM of 4 and 5 is 20. And that of 3 and 6 is 6.

We now get the denominators in the first bracket to be 20 and augment those in the second bracket to be 6 by

$$\left(\frac{21\times 5}{4\times 5}+\frac{11\times 4}{5\times 4}\right)-\left(\frac{7\times 2}{3\times 2}+\frac{19}{6}\right)$$

$$=\left(\frac{105}{20}+\frac{44}{20}\right)-\left(\frac{14}{6}+\frac{19}{6}\right)$$

In each bracket the denominators are the same now so we merely add the numerators.

$$\frac{149}{20}-\frac{33}{6}$$

The LCM of 20 and 6 is 60. This is from the fact that the prime factorizations of 6 and 20 are

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$$6 = 2 \times 3$$
$$20 = 2 \times 2 \times 5$$

The product of the biggest products of each prime number.

$$LCM(6, 20) = 2 \times 2 \times 3 \times 5 = 60.$$
$$\frac{149 \times 3}{20 \times 3} - \frac{33 \times 10}{6 \times 10} = \frac{447}{60} - \frac{330}{60}$$

Since the denominators are now equal we can just subtract the numerators.

$$\frac{447 - 330}{60} = \frac{117}{60}$$

To simplify note that 60 goes into 117 once and leaves a remainder of 57. So

$$\frac{117}{60} = 1\frac{57}{60}$$

We can further simplify using the GCF of 60 and 57. This is straightforwardly

$57 = 3 \times 19$ and $60 = 2 \times 2 \times 3 \times 5$.

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The least product of each prime number will be

$$GCF(57,60) = 2^0 \times 3^1 \times 5^0 \times 19^0$$
$$= 3$$

So we divide both the numerator and denominator by the $GCF = 3$.

$$1\frac{57}{60} = 1\frac{57 \div 3}{60 \div 3} = 1\frac{19}{20}$$

Our fraction is now reduced or simplified. We can also multiply and divide fractions. Let's say we desired to multiply or divide fractions. The rule is easy. For any $\frac{a}{b}$ and $\frac{c}{d}$, to multiply or divide we simply just multiply or divide the numerators and then the denominators, respectively.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}.$$

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Example 1Q

Evaluate

$$\frac{125}{25} \times \frac{1}{5} = \frac{125 \times 1}{5 \times 25} = \frac{125}{125} = 1.$$

Example 1R

Evaluate

$$\frac{125}{25} \div \frac{1}{5} = \frac{125}{25} \times \frac{5}{1} = \frac{125 \times 5}{25 \times 1} = \frac{125}{1} \times \frac{5}{25} = \frac{125}{1} \times \frac{1}{5} = \frac{125}{5} = 25.$$

Example 1S

Find the simplified value of

$$2\frac{4}{5} \times 1\frac{4}{21} \div \frac{5}{6} = \frac{14}{5} \times \frac{25}{21} \times \frac{6}{5} = \frac{14 \times 25 \times 6}{5 \times 21 \times 5} = \frac{84 \times 25}{21 \times 25} = \frac{84}{21} = 4.$$

Example 1T

Find a and b in the following

$$\frac{a}{3} = \frac{16}{24}$$

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and

$$\frac{15}{b} = \frac{5}{13}.$$

For

$$\frac{a}{3} = \frac{16}{24},$$
$$\frac{a \times 8}{3 \times 8} = \frac{16}{24}$$
$$\frac{8a}{24} = \frac{16}{24}$$
$$8a = 16$$
$$a = \frac{16}{8} = 2$$

Then for

$$\frac{15}{b} = \frac{5}{13},$$
$$\frac{15}{b} = \frac{5 \times 3}{13 \times 3} = \frac{15}{39}$$
$$b = 39.$$

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In chapter one we introduced decimal fractions and rounding off. What we did not consider, nonetheless, is that we can convert decimals into proper, improper and mixed fractions and vice versa.

As a means to transform a decimal fraction to a proper fraction we multiply by a number that makes our decimal an integer, multiples of ten are the easiest to work with.

Example 1U

1. Covert 7,65 to an improper fraction then subsequently a mixed fraction.
2. Convert 0.085 to a proper fraction.

Solution 1

We then divide and multiply by 100 as it makes 7,65 an integer, 765.

$$7.65 \times \frac{100}{100} = \frac{7.65 \times 100}{100} = \frac{765}{100} = \frac{765 \div 5}{100 \div 5} = \frac{153}{20}.$$

A common factor of 765 and 100 is five. This goes into a hundred twenty times and into 765 a hundred fifty three times. Twenty then goes into 153 seven times with a remainder of 13.

$$\frac{153}{20} = 7\frac{13}{20}.$$

Solution 2

Multiplying and dividing by 1000,

$$0.085 \times \frac{1000}{1000} = \frac{0.085 \times 1000}{1000} = \frac{85}{1000} = \frac{17}{200}.$$

Backwards, working out the decimal fraction from a proper, an improper and mixed fraction.

Example 1V

I. Convert $\frac{5}{12}$ to a decimal fraction.

II. Convert $\frac{65}{99}$ to a decimal fraction.

III. Convert $5\frac{3}{8}$ to a decimal fraction.

Solution I

$$\frac{5}{12} = 12\,\overline{)\,5}$$

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$$
\begin{array}{r}
0{,}4166 \\
12\;)\;5{,}^{5}0^{2}0^{8}0^{8}0
\end{array}
$$

These are remainders every time we divide.

Solution II

$$99\overline{)65}$$

99 cannot go into 65 so we place a zero comma on top and a comma after the 65 and add a zero but viewing it a 650 instead.

99 goes into 650 about 6 times so immediately after the comma our first decimal place we have 6. Leaving a remainder of 56.

$$
\begin{array}{r}
0{,}656\ldots \\
99\overline{)65} \\
-0 \\
\hline
65_0 \\
-594 \\
\hline
56_0 \\
-495 \\
\hline
65_0 \\
-594 \\
\hline
56_0
\end{array}
$$

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Solution III

$$5\frac{3}{8} = \frac{5 \times 8 + 3}{8} = \frac{43}{8}$$

$$\begin{array}{r}
5,375 \\
8\,\overline{)\,43} \\
-40 \\
\hline
3_0 \\
-24 \\
\hline
6_0 \\
-56 \\
\hline
4_0 \\
-40 \\
\hline
0
\end{array}$$

Our next idea is percentages. A percentage, %, is a proportion out of 100.

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Example 1W

$$6\% = \frac{6}{100}$$

$$10.5\ \% = \frac{10.5}{100}$$

$$150\ \% = \frac{150}{100}$$

25 % means 25 out of 100.

67 % means 67 out of 100.

127% means $\frac{127}{100}$

A ratio, on the other hand, is the comparison of two or more quantities.

$$12{:}\,30,$$

this means for 12 of one quantity we have 30 of another. The information on ratios can help us solve problems on proportions.

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Questions and Answers on Chapters 1:

1.

Round 20,54545... off to three decimal digits.

[1] 21,045

[2] 20,545

[3] 20,550

[4] 21,545

Question 1

20,54545 (Three Decimals Places)

20,54545 (Immediately follows the last decimal place needed but not 5, 6, 7, 8 or 9)

Solution 1

20,545 … [2]

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2.

Round 20,54545... off to two decimal digits.

[1] 20,60

[2] 21,04

[3] 20,55

[4] 21,05

Question 2

20,54545 (Two Decimals Places)

20,54545 (Immediately follows the last decimal place needed and it's one of 5, 6, 7, 8 or 9)

20,55 (we increase the last decimal digit we need, 4, by one to 5)

Solution 2

20,55 … [3]

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3. A shuttle and a bus leave Unisa at 9:00, heading for the FNB stadium. The bus arrives at the stadium 15 minutes after the shuttle. Suppose the bus takes y minutes to travel from Unisa to the stadium. An expression for the time it takes the shuttle to travel to the stadium is ...

[1] $(y - 15)$ minutes.

[2] $(9 + y - 15)$ minutes.

[3] $(y + 15)$ minutes.

[4] $(y + 15 - 9)$ minutes.

Solution 3

The bus arrives y minutes later but by then the shuttle had already been there for 15 minutes.

$(y - 15)$ minutes … [1]

4. Suppose I buy 10 shirts. Some cost R40 a shirt and others cost R45 a shirt. If the total cost is R435, how many of the R45 types of shirt did I buy?

[1] 10

[2] 7

[3] 5

[4] 3

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Solution 4

10 Shirts all together, that means if we denote the $R45$ shirts we bought by y and the $R40$ shirts by x, we have

$$x + y = 10$$

We then change the subject of the formula.

$$x = 10 - y \ldots (1)$$

Also

$$40x + 45y = 435 \ldots (2)$$

Because the number of shirts times the price of each gives us the total amount we've spent.

(1) Into (2) gives us

$$40\,(10 - y) + 45y = 435$$
$$400 - 40y + 45y = 435$$
$$5\text{y} = 435 - 400 = 35$$
$$y = \frac{35}{5} = 7 \ldots (3)$$

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(3) Into (1) we get

$$x = 10 - 7$$

$$x = 3.$$

We bought 7 shirts for $R45$ each and 3 shirts for $R40$ each.

5. Simplify the following expression as far as possible:

$$x(x-2) - 2(1-x^2)x - 4x$$

[1] $x^2 - 8x - 2x^3$

[2] $-x^3 + x^2 - 6x - 2$

[3] $-x^3 + x^2 - 4x - 4$

[4] $2x^3 + x^2 - 8x$

Solution 5

$$x(x - 2) - 2(1 - x^2)x - 4x$$

By the Distribution Law of multiplication over Addition $(a(b+c) = ab + ac)$,

$$x^2 - 2x - 2x + 2x^3 - 4x$$

$$2x^3 + x^2 - 8x \ldots [4]$$

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6. A mother divides an amount of money among her three children, Kagiso, Dikeledi and Thabo. Kagiso gets twice as much as his sister, Dikeledi, and Dikeledi gets R100 less than Thabo. Suppose Thabo gets x rand. How much does Kagiso get in terms of x?

[1] $\left(\frac{x}{2}+100\right)$ rand

[2] $\left(\frac{100-x}{2}\right)$ rand

[3] $\frac{2(x+100)}{3}$ rand

[4] $2(x-100)$ rand

Solution 6

Thabo = T = x

Dikeledi = D = 100 less than x = $x - 100$

Kagiso = K = 2 times as much as $(x - 100)$ = R$2(x - 100)$… [4]

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7. Simplify the following expression as far as possible:

$$\sqrt{b^8} \cdot \sqrt{b^{16}}$$

[1] b^{32}

[2] b^{8}

[3] b^{12}

[4] b^{64}

Solution 7

$$\sqrt{b^8} \times \sqrt{b^{16}} = \sqrt{b^8 \times b^{16}} = \sqrt{b^{8+16}} = \sqrt{b^{24}} = b^{\frac{24}{2}} = b^{12} \ldots [3]$$

8. Simplify the following expression as far as possible:

$$a^{x+3} \cdot a^{-x-2}$$

[1] a

[2] a^2

[3] a^{-x^2-6}

[4] $2a$

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Solution 8

$$a^{x+3} \times a^{-x-2} \times a^0 \times b^0 = a^{x+3-x-2+0} \times b^0 = a$$

Because any real number to the power of zero is one.

9.

Determine the LCM of the following three terms:

$$6x^3, 8x^2y^2 \text{ and } 12xy^5$$

[1] $2xy^5$

[2] $24x^3y^5$

[3] $576xy$

[4] $24x^6y^7$

Solution 9

For each variable the LCM is the one with the highest power. That of the coefficients is simply as we've learned to calculate it using prime factorizations.

$$LCM(6, 8, 12) = 24$$
$$LCM(x^3, x^2, x) = x^3$$
$$LCM(y^0, y^2, y^5) = y^5$$

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So then the
$$LCM(6x^3, 8x^2y^2, 12xy^5) = LCM(6, 8, 12) \times LCM(x^3, x^2, x) \times LCM(y^0, y^2, y^5)$$
$$= 24x^3y^5 \ldots [2]$$

10. Simplify the following expression as far as possible:

$$\frac{1}{4x^2} + \frac{2}{5x}$$

[1] $\dfrac{3}{4x^2 + 5x}$

[2] $\dfrac{5x}{8x^2}$

[3] $\dfrac{5 + 8x}{20x^2}$

[4] $\dfrac{3}{9x^2}$

Solution 10

This is a fraction and remember we can only add if the denominators are equal, to get the right common denominator we use the $LCM(4x^2, 5x)$.

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The result of which will be

$$LCM(4,5) = 20$$
$$LCM(x^2, x) = x^2$$
$$LCM(4x^2, 5x) = 20x^2.$$

Furthermore, to get the denominators the same we had to

$$\frac{1\times5}{20x^2} + \frac{2\times4x}{20x^2} = \frac{5+8x}{20x^2} \ldots[3]$$

11. A blended fruit juice contains three main ingredients, namely mango juice, orange juice and distilled water, mixed in the ratio 4 : 5 : 1, respectively. How many litres of orange juice are needed to make 25 litres of this fruit juice?

[1] 2,5 litres

[2] 5 litres

[3] 10 litres

[4] 12,5 litres

Solution 11

$$1:4:5$$
$$DW:MJ:OJ$$

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We have

$$\frac{25}{1+4+5} = 2.5 \; Per \; Portion.$$

We, nevertheless, need five portions of Orange Juice.

So

$$5 \times 2.5 = 12.5 \text{ Litres} \ldots [4]$$

Portions We Need × Litres Per Portion

12. Bargains-4-U sells a washing machine for R1 320, excluding VAT. We assume that VAT is 14%. If you pay cash, the company offers a 5% discount. How much will you save if you buy this washing machine in cash?

[1] R75,24

[2] R66,00

[3] R135,43

[4] R118,80

Solution 12

$$\text{Price}_{IncludingVAT} = R\ 1320 \times \frac{114}{100} = R1504.80$$

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$$\text{Discount Value} = R1504{,}80 \times \frac{5}{100} = R75{,}24\ldots[1]$$

13.

Simplify the following expression as far as possible without using a calculator:

$$\frac{4}{5} - \frac{5}{6} + \frac{1}{4}$$

[1] 0

[2] $-\frac{1}{5}$

[3] $\frac{13}{60}$

[4] $1\frac{1}{20}$

Solution 13

The prime factorizations are

$$6 = 2 \times 3 \times 5^0$$
$$4 = 2 \times 2 \times 3^0 \times 5^0$$
$$5 = 2^0 \times 3^0 \times 5$$

$$LCM(5, 6, 4) = (2 \times 2) \times 3 \times 5 = 60.$$

And so

$$\frac{4 \times 12}{60} - \frac{5 \times 10}{60} + \frac{1 \times 15}{60} = \frac{48}{60} - \frac{50}{60} + \frac{15}{60} = \frac{13}{60}.$$

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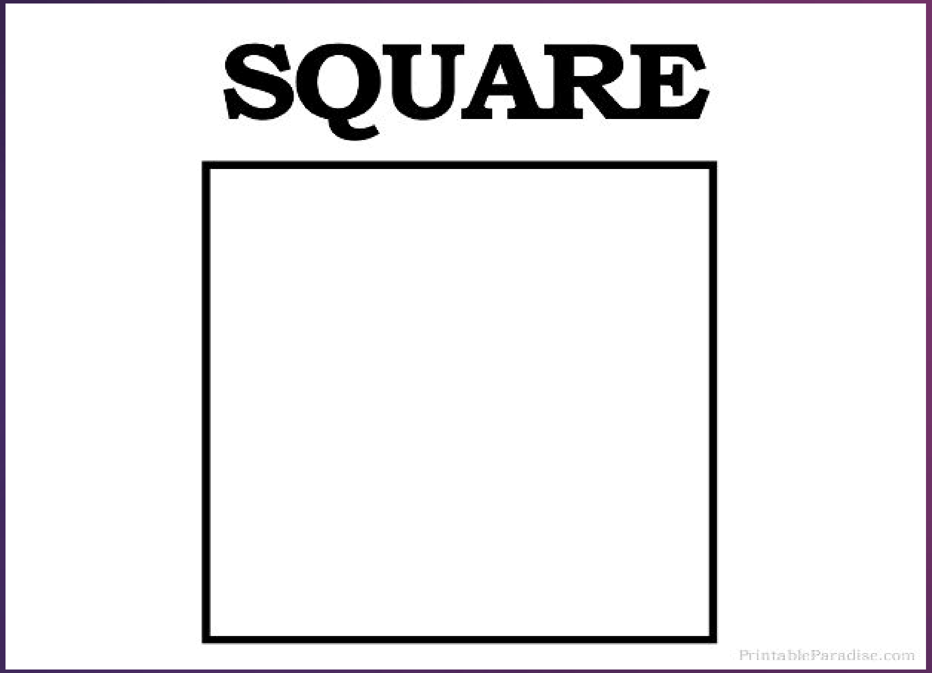
MEASUREMENTS

There are four shapes we look at known as regular figures. These include - squares, rectangles, triangles and circles. These regular figures all have a standard way of measuring the Perimeter (the distance of the outline of a shape,) the Area (surface size) and Volume (a measure of space.)



Perimeter

$P = l + l + l + l = 4l$

The l's are the lengths of the sides.

Area

$A = l \times l = l^2$

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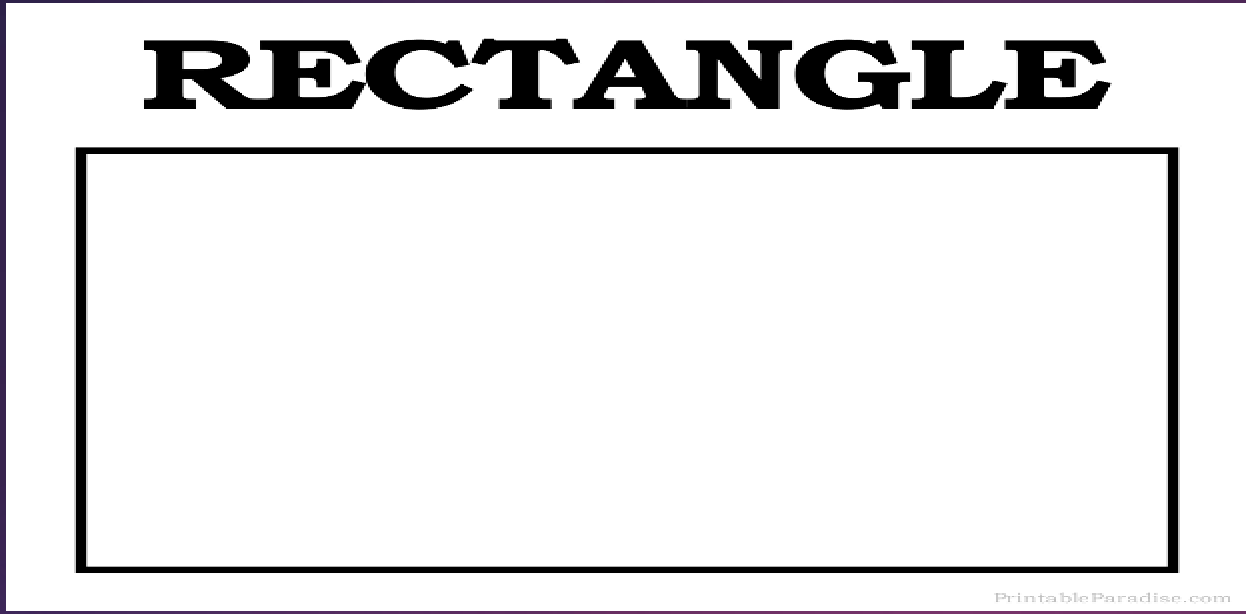

Perimeter

$$P = 2l + 2W = 2(l + W)$$

Where l and W are the length and width , respectively.

Area

$$A = l \times W$$

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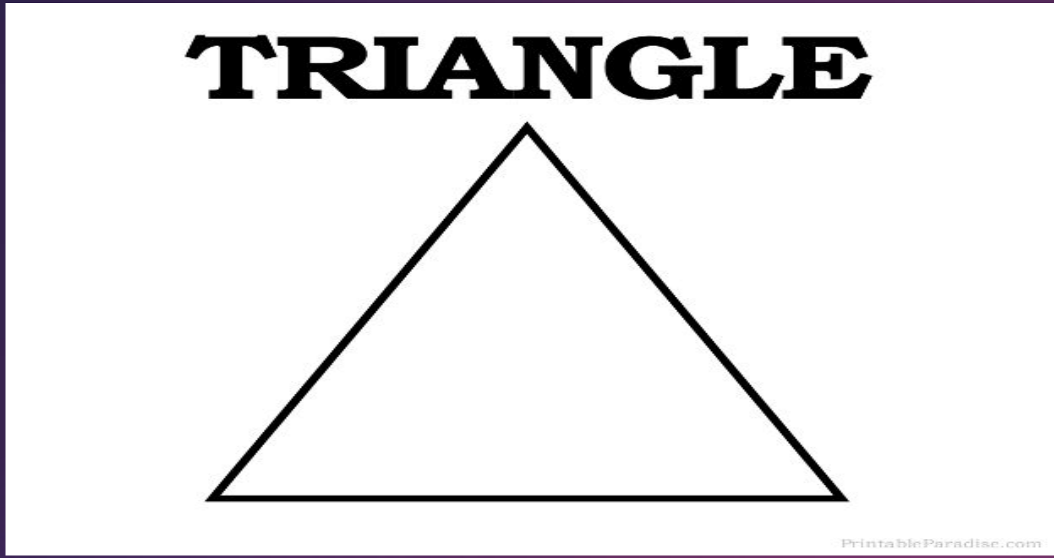

Perimeter

$P = a + b + c$

a, b and c are the sides of the triangle.

Area

$A = \frac{1}{2} Base \times Height$

The base is the side you choose as your horizontal (flat) side. The height is the line from the highest vertex going down to meet the base at a ninety degree (right) angle.

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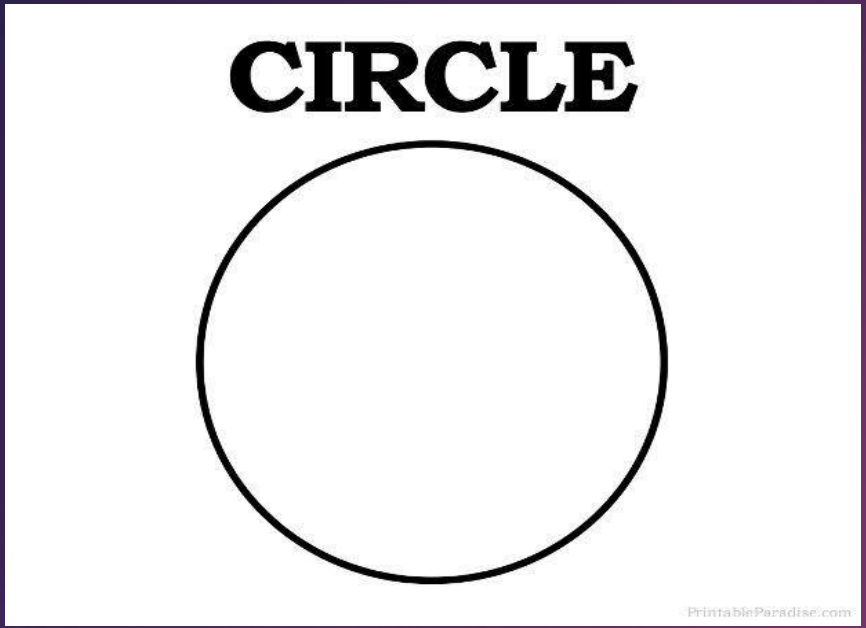

Circumference

The perimeter of a circle is called a circumference. The radius (the line from the centre of the circle to the circumference) is "r." The diameter (the line from one point on the circumference through the centre to another point on the circumference is "d."

$$C = 2\pi r$$

$$r = \frac{d}{2}$$

Therefore, $C = 2\pi\left(\frac{d}{2}\right) = \pi d.$

Area

$$A = \pi r^2$$

And therefore

$$A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$$

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To enable us to work-out or value the circumference or area we will need a standard measure of length or distance, this is widely known as metres, and denoted as "m."

More often than not, the metres have a prefix that tells us how many of these metres we have (for example, "centi," to get centimetres. Shortly known as "cm.") The base unit for distance is metres together with a prefix from the following table:

The following table will give us a good indication of the SI sytem:

Number	Power	Common name	SI name	SI abbreviation
10	10^1	ten	deca -	D
100	10^2	hundred	hecta -	h
1 000	10^3	thousand	kilo -	k
1 000 000	10^6	million	mega -	M
1 000 000 000	10^9	milliard*	giga -	G
1 000 000 000 000	10^{12}	billion	tera -	T
0,1	10^{-1}	tenth	deci -	d
0,01	10^{-2}	hundredth	centi -	c
0,001	10^{-3}	thousandth	milli -	m
0,000001	10^{-6}	millionth	micro -	μ
0,000000001	10^{-9}	milliardth	nano -	n
0,000000000001	10^{-12}	billionth	pico -	p

*(American word for milliard is billion.)

These are named the SI units of measurement for length. We can always convert between these measurements of length as in the equations that follow.

The standard unit for length is the metre. The SI abbreviation for metre is m. Therefore we write 10 metres as 10 m.

Lengths in the SI system:

10 millimetres	(mm)	=	1 centimetre	(cm)
10 centimetres	(cm)	=	1 decimetre	(dm)
10 decimetres	(dm)	=	1 metre	(m)
10 metres	(m)	=	1 decametre	(dam)
10 decametres	(dam)	=	1 hectometre	(hm)
10 hectometres	(hm)	=	1 kilometre	(km)

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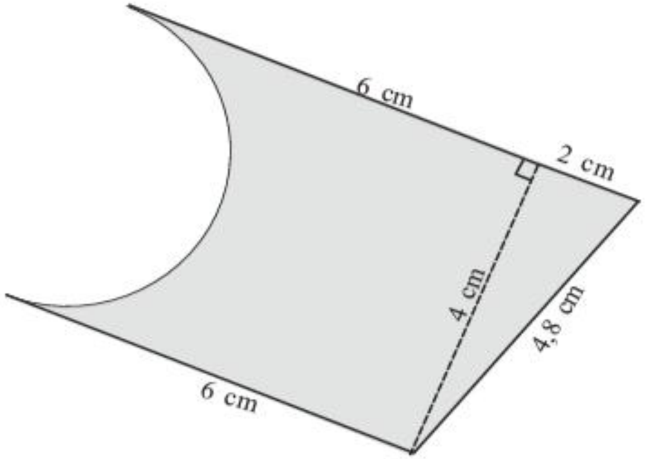
Example 1X

Consider the diagram below. Measurements are indicated on the diagram. The semi-circle fits perfectly into the shorter side of the rectangle.



Calculate the perimeter of the area that is shaded in the diagram.

[1] 31,37 cm

[2] 21,72 cm

[3] 15,43 cm

[4] 25,08 cm

*This is what is known as a composite figure due to it being a combination of different regular figures.

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From the rectangle and triangle we only have two sides each.

$$P_R = 2l = 2(6\text{cm}) = 12\text{ cm}$$

$$P_T = a + b = 2\text{ cm} + 4{,}8\text{ cm} = 6{,}8\text{ cm}$$

The circumference of the full circle will be

$$P_{FC} = \pi d = (4\text{cm})\pi = 4\pi\text{ cm},$$

while that of the semi-circle

$$P_{SC} = \frac{1}{2}P_{FC} = \left(\frac{1}{2}\right)4\pi\text{ cm} = 2\pi\text{ cm}.$$

The perimeter of the shaded area is then the sum,

$$P_{SA} = P_{SC} + P_T + P_R$$

$$P_{SA} = 2\pi\text{ cm} + 6{,}8\text{ cm} + 12\text{ cm}$$

$$= 25{,}08318531\text{ cm}\ldots[4]$$

If we wanted to convert this value to metres, m, we'd take the following steps:

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Aware that for 1 cm we have $\frac{1}{100}$ m, expressly,

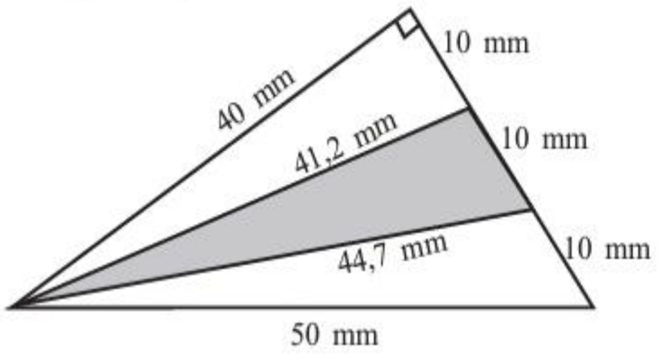
$$1 \text{ cm} : 0{,}01 \text{ m}.$$

For this reason then

$$P_{SA} = 25{,}083 \ldots (0.01\text{m}) = 0.25083\text{m}$$

Example 1Y

Consider the diagram that has one right angle below. Measurements are indicated on the diagram.



Calculate the area of the shaded part in the diagram.

[1] 200 mm^2

[2] 206 mm^2

[3] 223,5 mm^2

[4] 194 mm^2

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Area is measured in square units because of the two dimensions. Particularly: $\mathrm{cm}^2, \mathrm{m}^2, \mathrm{km}^2$ and so on.

We can calculate the area of the second triangle and from it subtract the area of the first to obtain the area of the shaded region.

$$A_S = A_2 - A_1 = \frac{1}{2}\, b \times h - \frac{1}{2}\, b \times h$$

$$A_S = \frac{1}{2}\,(40\ \mathrm{mm} \times 20\ \mathrm{mm}) - \frac{1}{2}\,(40\ \mathrm{mm} \times 10\ \mathrm{mm})$$

$$= 400\ \mathrm{mm}^2 - 200\ \mathrm{mm}^2$$

$$= 200\ \mathrm{mm}^2 \ldots [1]$$

$$= 200\ (0.1\ \mathrm{cm})^2$$

$$= 2\ \mathrm{cm}^2$$

The unit of measure for volume is cubed length units, that is to say, to the power of three. But we also have litres for volume and we can convert between the two at will.

$$1000\ \mathrm{cm}^3 = 1l$$

$$1\mathrm{m}^3 = 1000l$$

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Volume counts how many $1\ \mathrm{m}^3, 1\ \mathrm{cm}^3$ or $1\ \mathrm{mm}^3$ and so on can we fit into a specific space. The formula of which is

$$V = l \times w \times h \text{ (for prisms and cubes)}$$

$$V = \pi r^2 h \text{ (for cylinders)}$$

l, w, h and r are length, width, height and radius, respectively.

Example 1Z

In this example we need to find both the volume of a cylinder and a prism.

$$V_{\text{Metal}} = V_{\text{Cylinder}} - V_{\text{Prism}}$$
$$V_{\text{Cylinder}} = \pi r^2 h$$
$$V_{\text{Cylinder}} = (6\ \mathrm{mm})^2 \times 30\ \mathrm{mm} \times \pi = 1080\pi\ \mathrm{mm}^3$$
$$V_{\text{Prism}} = 5\ \mathrm{mm} \times 5\ \mathrm{mm} \times 30\ \mathrm{mm}$$
$$V_{\text{Prism}} = 750\ \mathrm{mm}^3$$
$$V_{\text{Metal}} = 1080\pi\ \mathrm{mm}^3 - 750\ \mathrm{mm}^3$$

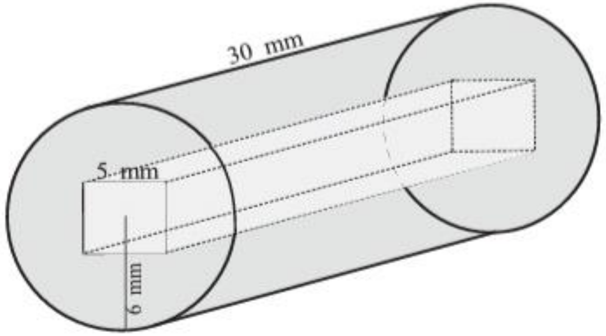
$$V_{\text{Metal}} = 2642{,}920066\ \mathrm{mm}^3$$

$$V_{\text{Metal}} = 2642{,}920066 \times (0.1\ \text{cm})^3$$
$$= 2{,}64292\ \text{cm}^3 \ldots [3]$$

Since

$$1000\ \text{cm}^3 = 1l \rightarrow 1\ \text{cm}^3 = \frac{1}{1000}\ l \rightarrow 2{,}64292\ \text{cm}^3 = \frac{2{,}64292}{1000}\ l = 0{,}002643\ l$$

Refer to the sketch below. A cylindrical piece of steel, which is 30 mm long and has a radius of 6 mm, has a square hole right through it in its length. The hole is in the centre of the rod. The sides of the square are each 5 mm.



Calculate the volume of metal that this small object contains.

[1] $98{,}2\ \text{mm}^3$

[2] $264{,}29\ \text{mm}^3$

[3] $2{,}64\ \text{cm}^3$

[4] $2{,}49\ \text{cm}^3$

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EXPRESSIONS, EQUATIONS AND FORMULAS

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A combination of numbers, operators and variables like

$$3x + 2y - 6$$

is an expression. If we, however, equate this to a value or another variables it then becomes an equation. Therefore,

$$3x + 2y - 6 = 0$$

is an equation.

A formula is an equation illustrating the relationship between two or more variables, usually of a scientific phenomenon like temperature, velocity or acceleration.

We solve equations by solving for a specific unknown (finding the value of a specific variable.) As examples, consider the following:

Example 5A

$$x + 6 = 26$$

$$x + 6 - 6 = 26 - 6$$

$$x = 20$$

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Example 1ZA

$$\frac{y}{13} + 2 = 3$$

$$13 \times \left(\frac{y}{13} + 2\right) = 3 \times 13$$

$$y + 26 = 39$$

$$y + 26 - 26 = 39 - 26$$

$$y = 13$$

Example 1ZB

$$25z^2 - 1 = 0$$

$$25z^2 = 1$$

$$z^2 = \frac{1}{25} = \frac{1}{5^2}$$

$$\sqrt{z^2} = \sqrt{\frac{1}{5^2}} = \frac{\sqrt{1}}{\sqrt{5^2}}$$

$$z = \pm\frac{1}{5}$$

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Example 1ZC

$$\sqrt{2x^2+1} = 3$$
$$\left(\sqrt{2x^2+1}\right)^2 = 3^2$$
$$2x^2+1 = 3^2$$
$$x^2 = \frac{9-1}{2} = 4$$
$$\sqrt{x^2} = \sqrt{4}$$
$$x = \pm 4$$

Example 1ZD

Consider

$$400\,(1+i)^t = 2000.$$

Making t the subject of the formula means getting t alone on one side. Which brings us back to the $\log$. We want t alone so

$$(1+i)^t = \frac{2000}{400}$$

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$$(1+i)^t = 5$$

Right here is where we introduce the logarithm, we $\log$ both sides of the equation.

$$\log(1+i)^t = \log 5$$

From the property of $\log$s introduced in chapter two.

$$t \times \log\,(1+i) = \log 5$$

$$t = \frac{\log 5}{\log(1+i)}$$

Now t is the subject of the formula.

Example 1ZE

Make a the subject of the formula of the equation,

$$s = ut + \frac{1}{2}at^2,$$

from physics calculating distance, s, using initial velocity (u), acceleration -a- and time, t.

$$s - ut = \frac{1}{2}at^2$$

$$2 \times (s - ut) = 2 \times \frac{1}{2} at^2$$
$$at^2 = 2(s - ut)$$
$$a = \frac{2(s - ut)}{t^2}$$

Expressions can also be simplified.

Example 1ZF

$$ab(a - b) - b(c^2 - ab) - (a^2 - c^2)b$$
$$= a^2b - ab^2 - bc^2 + ab^2 - a^2b + bc^2$$
$$= -ab^2 - bc^2 + ab^2 + bc^2$$
$$= -bc^2 + bc^2$$
$$= 0$$

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SIGNS, NOTATIONS AND COUNTING RULES

The sign $<$ reads less than from left to right. $\leq$ means less than or equal. The reverse $\geq$ reads greater than or equal and $>$ is greater than.

The notation for summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n.$$

The are a few rules to counting combinations and permutations. Suppose we want to order n numbers. We can get the first number from n different options. However, once we've picked a first one we are left with $n - 1$ options for the second number. And so on. The total number of ways to order this is n factorial. The factorial is symbolized by the exclamation mark

$$n \times (n - 1) \times \cdots \times 1 = n!$$

To explain the concept of permutations I'll use this example.

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Example 1ZG

Seven horses are in a race. How many different ways can the race end.

The first place can be taken by any of the seven horses. We now have only six places and six horses left. Second place can be claimed by the six horses left.

The number of ways the horses can complete the race is

$$7! = 5040.$$

The point here is that the order is important if we switch any two of these the we have a completely set. If this is the case we call it a permutation. For permutations we have a formula.

The number of permutations of x objects out of n objects is

$$mPx = \frac{m!}{(m - x)!}$$

So for the previous example if would be

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$$m = 7$$
$$x = 7$$
$$\frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1} = 7! = 5040$$

A combination is when the order does not matter. This means in any order you get the same thing like choosing a team from a group of people. It does not matter who you pick first the group will be the same unless you assign title by order.

The formula of combinations means removing repetitions. For instance, picking three deserts from ten options. You can always pick them in a different order. However, picking the first, the third and sixth desert and picking the sixth, the first and third will the same thing. Even choosing the first, sixth and third. So we need to remove the repetitions.

For combinations

$$mCx = \frac{m!}{(m-x)!\,x!}.$$

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Chapter Two

COLLECTION, PRESENTATION AND DESCRIPTION OF DATA

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Data is categorized information much of which is used for statistical purposes. We begin this chapter with sampling methods. A sample is a representative fraction of the population (the set of all elements or items being studied) which we can use to make conclusions about parameters of the population group. Sampling methods attempt to get as random a sample as possible. First on our list of methods we have

Simple Random Sampling

If we have a population and order the population units numerical then based on a random choice of numbers we could pick a random sample much like in the following examples.

Example 3A

Let the numbers from one to thirty represent accounts as in the image on the next slide. Suppose that the following

$$22; 17; 83; 57; 27; 54; 19; 51; 39; 59; 84; 20$$

are computer generated random numbers.

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A printing company, Printapage, has 30 clients with the following outstanding balances (in rand).

Account No.	Balance	Account No.	Balance
1	25	16	0
2	0	17	102
3	605	18	215
4	1 010	19	429
5	527	20	197
6	34	21	159
7	245	22	279
8	59	23	115
9	667	24	27
10	403	25	27
11	918	26	291
12	801	27	16
13	227	28	0
14	0	29	402
15	47	30	570

The point of simple random sampling is to use these random numbers to pick a sample of a particular size. We begin by removing all numbers outside the interval of one and thirty so we are left with

$$22; 17; 27; 19; 20.$$

For a simple random sample of five sample units these will be our accounts

$$22; 17; 27; 19; 20$$

with balances

$$279; 102; 16; 429; 197.$$

Example 3B

A city has eight hundred fifty three medical doctors. Of which a random sample of fifteen has to be chosen with the following random numbers:

$$253; 872; 489; 230; 313; 287; 875; 844; 348; 134$$
$$794; 064; 437; 041; 355; 898; 910; 207; 194; 804$$

From the random numbers remove the number not between one and eight hundred fifty three. This is because if we had nine hundred it would not represent anything from our population because we have only eight hundred fifty three medical practitioners. So our random number without the outside values

$$253; 489; 230; 313; 287; 844; 348; 134$$
$$794; 064; 437; 041; 355; 207; 194; 804$$

Then the first fifteen elements represent our sample that would be

$$253; 489; 230; 313; 287; 844; 348; 134$$
$$794; 064; 437; 041; 355; 207; 194$$

Stratified Random Sampling

This method of sampling is much like simple random sampling but within groups into which we divide our population.

Example 3C

A club has twenty five students and ten faculty members. Using the random numbers

$$14; 78; 85; 19; 10; 34; 08; 60; 25; 36; 14; 06; 16; 04; 83$$

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pick a random sample of size six.

Our data is divided into two stratums, students and faculty members. The largest one is the students stratum which contains twenty five elements. From our random numbers any number greater than twenty five must be discarded. Therefore we have

$$14; 19; 10; 08; 25; 14; 06; 16; 04 \ldots (1)$$

Out of a total number of thirty five members of the club, twenty five are students and the rest, ten, are faculty members. By proportion

$$\frac{25}{35} \times 6 = 4{,}28 \approx 4$$

must be students and

$$\frac{10}{35} \times 6 = 1{,}72 \approx 2$$

will come from the faculty members pool. Using the random numbers …(1) we choose the first

$$14; 19; 10; 08.$$

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Continuing from right where we ended off we have

$$06; 04$$

because our second stratum has only items from one to ten so our two faculty members will be numbers

$$06; 04.$$

Our complete sample of six items will then be

$$14; 19; 10; 08; 06; 04.$$

The point of stratified random sampling is to first divide our populations into group we know of in our population. Work our the proportion that must come from each group. Then we can make our random selection.

Data comes as quantitative (discrete or continuous) or qualitative. Continuous quantitative comes when the variable itself is continuous like height but if the variable itself is discrete the we have discrete quantitative data. Qualitative data is when our data is not a comparison of numbers.

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We'd now like to represent our data as a histogram and then a pie chart using the Radial Data

Radial is happy that its sample is representative. The sample elements are (in thousands)

61	38	19	58	66	64	72	66	64	75
42	24	77	70	46	69	45	46	59	16
59	72	46	50	37	78	66	75	66	67
98	64	64	72	59	88	75	67	45	61
61	77	29	26	62	80	22	83	53	51
82	16	78	34	70	50	69	54	78	77
45	62	45	58	90	86	62	50	56	58
51	32	86	40	62	70	40	67	80	66
14	54	51	54	67	64	69	51	48	72
32	46	22	30	61	74	74	62	64	75

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1. We find the range, R, of the data.

$$R = \text{Max} - \text{Min}$$

$$R = 98 - 14 = 84$$

2. Choose the number of elements you want shared between these intervals(make those between five and ten.) The you divide your range by your choice.

$$\frac{R}{10} = \frac{84}{10} = 8{,}4 \approx 8 \text{ Intervals}$$

3. Is determining how long your intervals will be that is the range divided by the number of intervals. It must be a whole number.

$$\frac{R}{8} = \frac{84}{8} = 10{,}5 \approx 11.$$

4. Determining the limits of the intervals. If we shift the lower number of the first interval to make it slightly lower then the issue of overlapping numbers is avoided. If we begin at

$$14 - 0{,}5 = 13{,}5$$

and add our eleven length every time we will have 8 intervals of length eleven then we can start counting how many numbers fall within each interval. The rest of chapters two and three are quite well explained in the guides these are also less subjective topics. This means they are not subject to much interpretation.

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Chapter Three

INDICES AND TRANSFORMATIONS

See your guides and email me if there is anything you do not understand.

Chapter Four

STRAIGHT LINES AND LINEAR EQUATIONS

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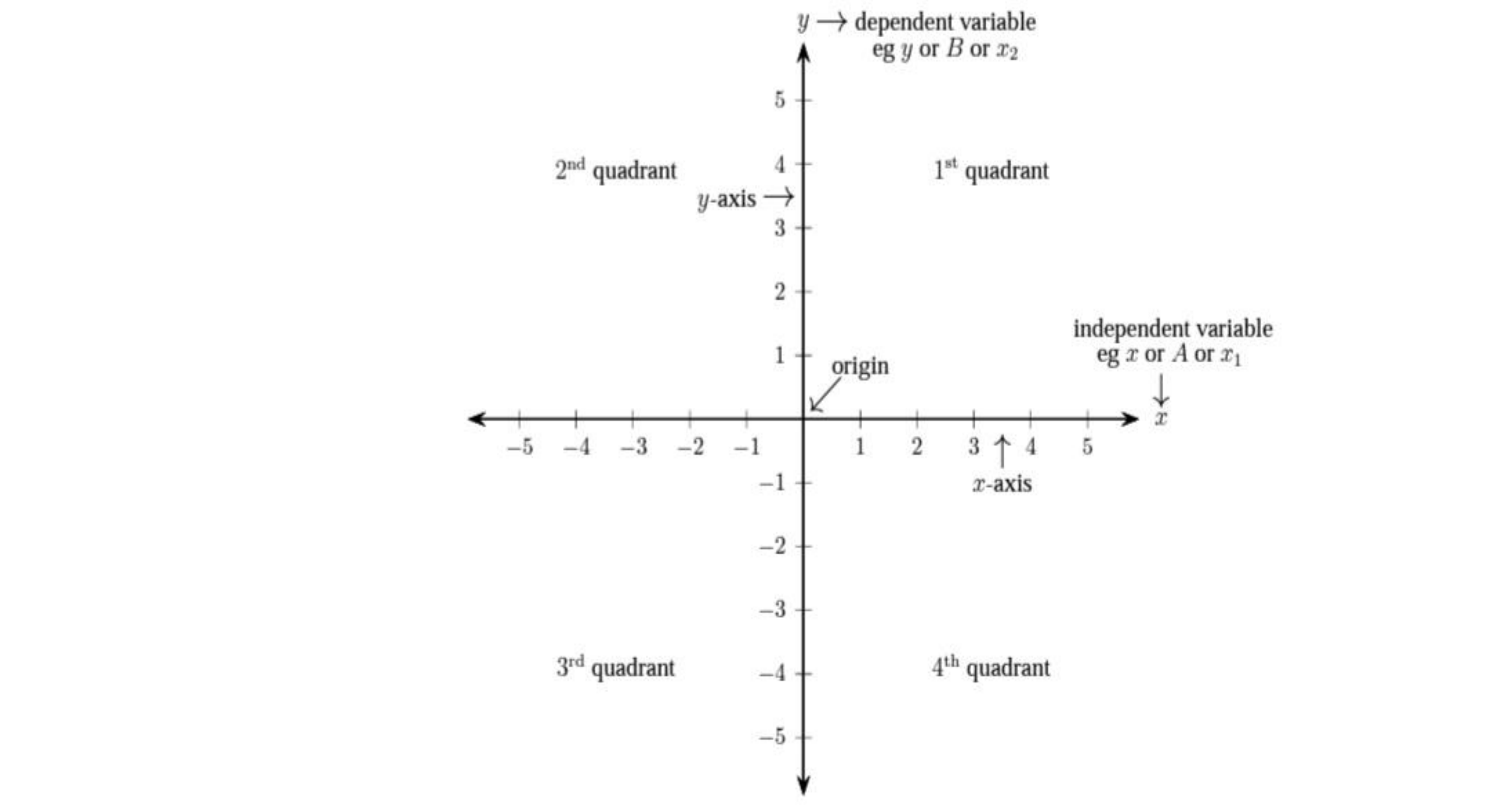
Figure 4A



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Imagine we had two number lines. Call them y and x. Suppose we intercept these number lines at their zeros. Consequently with four quadrants, we have what is titled the (x, y) co-ordinate system of axis.

The first quadrant being the top right one. Second the top left one. The bottom left one is declared the third quadrant. Lastly, the fourth quadrant is on the bottom right. This is illustrated in Figure 6A.

On this plane we get points which translate to a pair of x and y, written (x, y). For the point $(2, -6)$, we recognize that $x = 2$ and $y = -6$. Because number lines are continuous (there is a value at every point on it, even between the integers) we also have fraction values of x and that of y, that is to say, $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

It is worth noting that we only need two of the points on the (x, y)-plane to draw a straight line going through each point similar to the line in the ensuing (following) Example.

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Example 4A

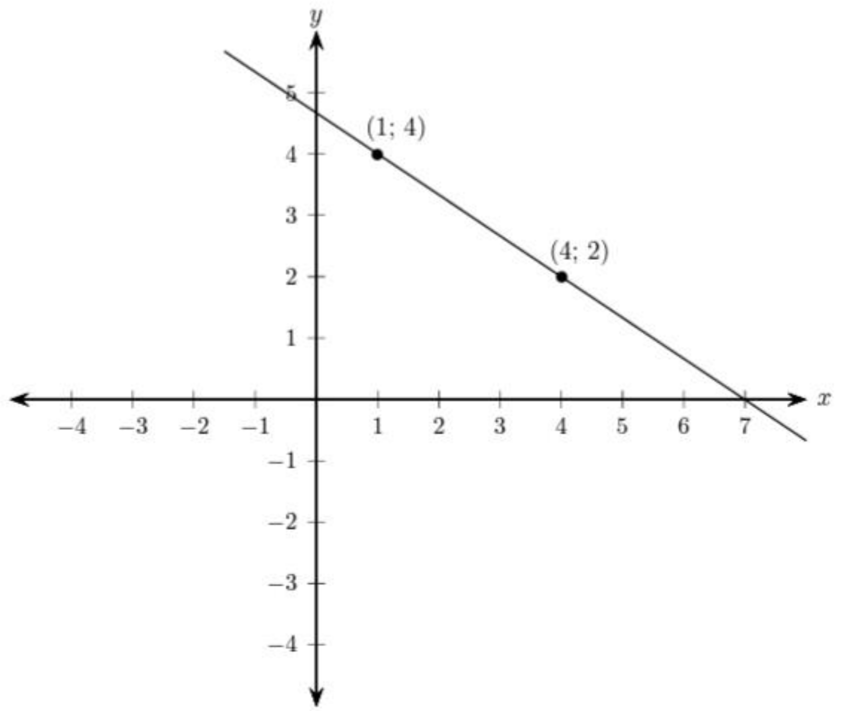
Plot the straight line that passes through the points $(1, 4)$ and $(4, 2)$.

Straight lines posses, however, a slope (incline or decline) which may be any real value negative or positive. This slope is known as the line's gradient and it is important for the equation that models an algebraic representation of a straight line also known as a linear function. The formula for the gradient, g, is

$$g = \frac{y_0 - y_1}{x_0 - x_1}.$$

Where (x_0, y_0) is one point and (x_1, y_1) is another. It is not important which one you choose to be which.

For our Example 6A the line looks something like following graph on the next slide.



The formula for the straight line is then

$$y = gx + b.$$

Where g is the gradient and b the y-intercept.

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Example 4B

Use Example 6A to find the equation of the line.

$$g = \frac{y_0 - y_1}{x_0 - x_1} = \frac{4-2}{1-4} = -\frac{2}{3}$$

Since we don't know the y-intercept we can use one of the points on the line to find b.

$$y = -\frac{2}{3}x + b \ldots (1)$$

Substituting $(1,4)$ into …(1)

$$4 = -\frac{2}{3}(1) + b$$

$$4 + \frac{2}{3} = b$$

$$\frac{12}{3} + \frac{2}{3} = b$$

$$b = \frac{14}{3}$$

Therefore,

$$y = -\frac{2}{3}x + \frac{14}{3} \cdots 3y = -2x + 14$$

Example 4C

With the formula in Example 6B we can now find both intercepts, that of y and of x.

y-intercept is where $x = 0$.

$$y = -\frac{2}{3}(0) + \frac{14}{3} = \frac{14}{3} = b$$

We could have just said the y-intercept is b.

x-intercept is where $y = 0$.

$$3(0) = -2x + 14$$
$$-14 = -2x$$
$$x = \frac{-14}{-2} = 7.$$

The intercepts are also points. The y-intercept is the point

$$(0, y) = (0, b) = (0, \frac{14}{3})$$

and x-intercept represents the point

$$(x, 0) = (7, 0).$$

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If the gradient, g, is less than zero (negative) then the line is decreasing (descending). But if the gradient, g, is greater than zero (positive) our straight line is increasing (ascending.)

Example 4D

Since the line in Example 6A has a gradient, g,

$$g = -\frac{2}{3}$$

which is less than zero or negative. Our line is, therefore, descending or decreasing. This is true from the picture as well.

Consider

$$y = gx + b$$

with $g = a$.

That is to say,

$$y = ax + b.$$

Well then we have

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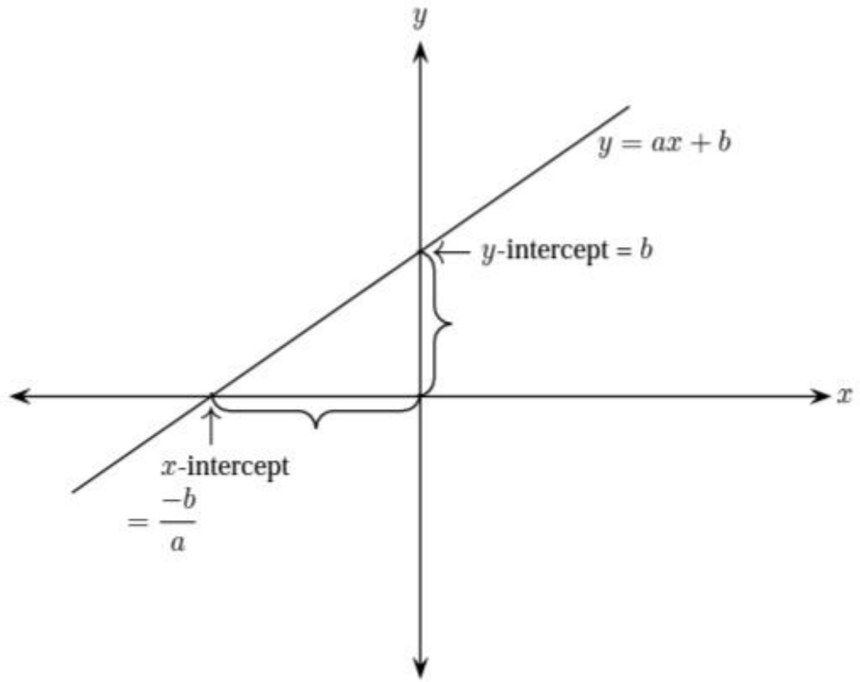
The following is true for straight ling graphs.

Thus, we can conclude that the line cuts the axes at the points $(0; b)$ and $\left(-\frac{b}{a}; 0\right)$ as shown in figure 4.2.2.



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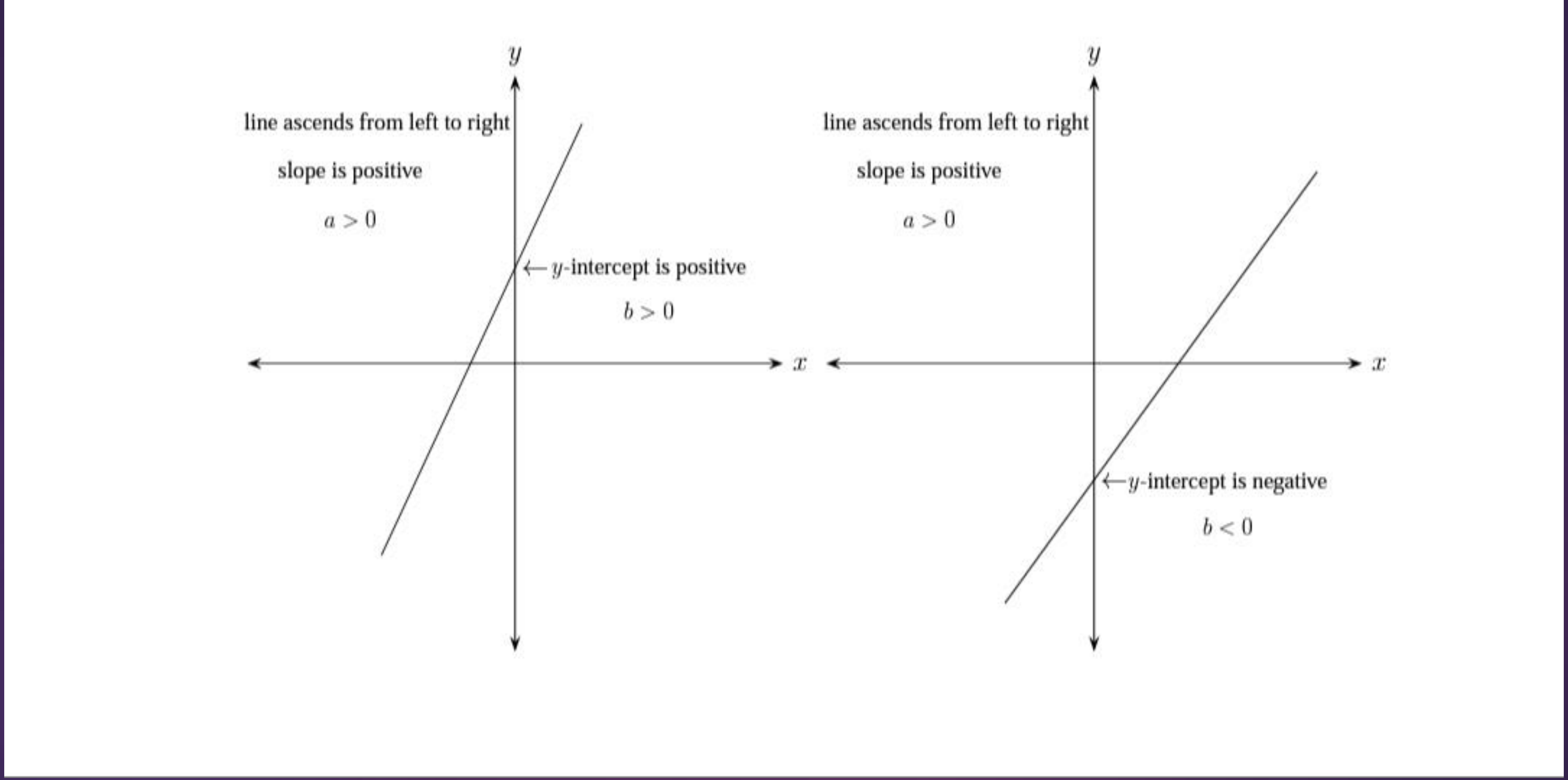
The direction of the lines determine the sign of its gradient. This presents a positing slope.



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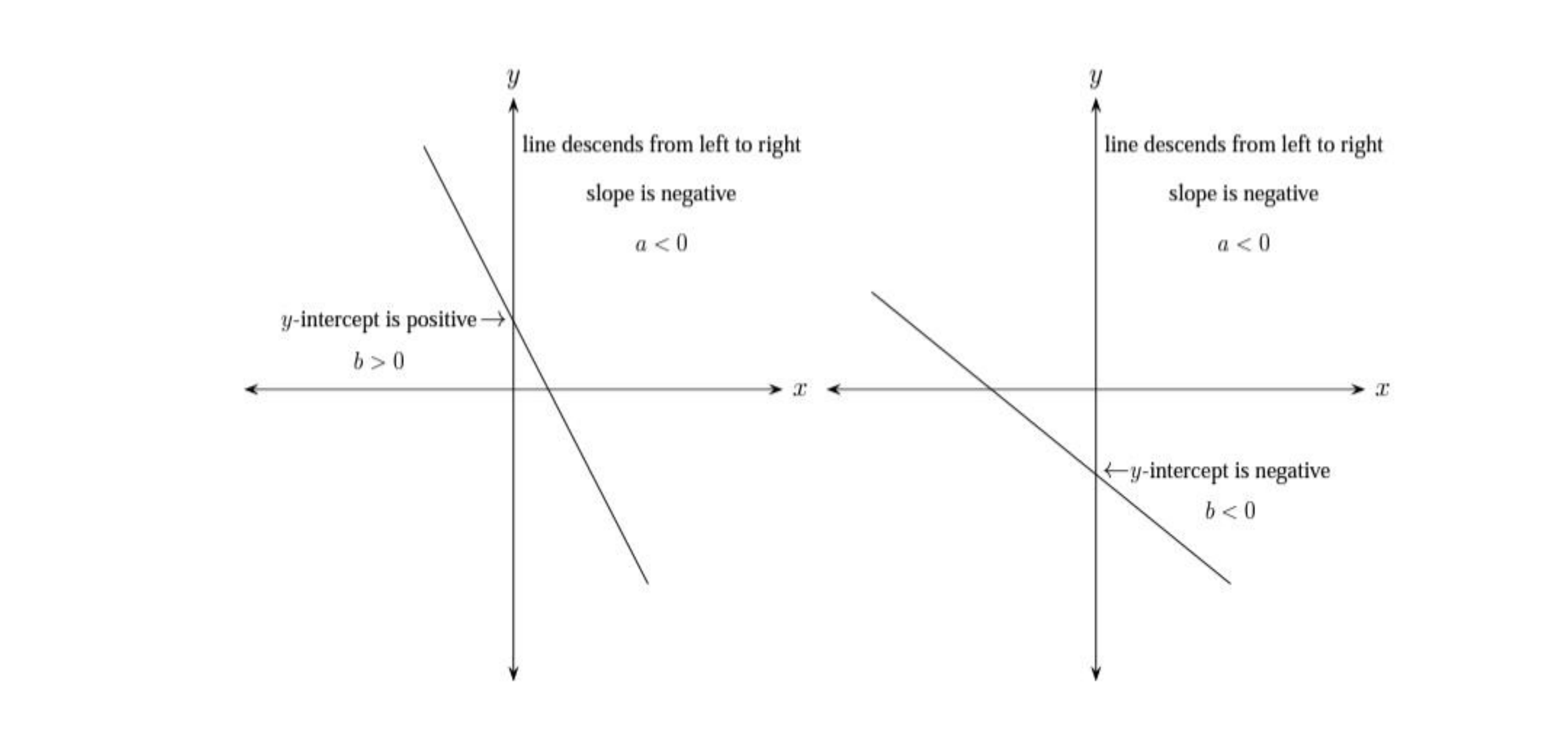
For a negating slope the graph sits as portrayed in the image that follow.



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When we have horizontal lines it means the slope is zero.

$$y = a(0) + b = b$$
$$y = b.$$

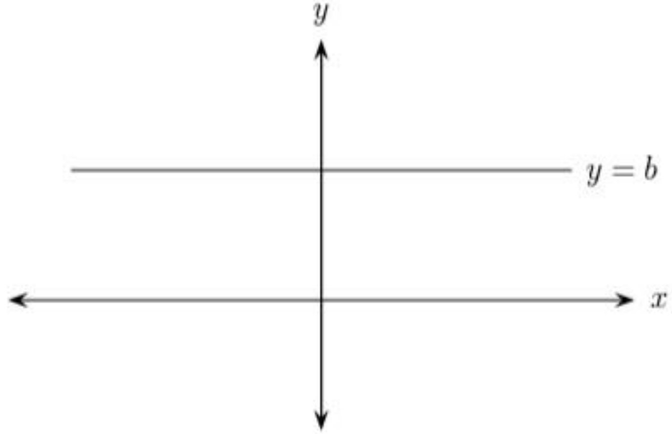
A sketch of which

- **The case of a zero valued slope, this is $a = 0$.**

  What does this mean? Looking at our expression for a, that is

  $$\frac{y_2 - y_1}{x_2 - x_1},$$

  we see that this can only be the case if $y_2 = y_1$. That is, if the function values are the same and y is not dependent on x, it is in fact a constant. This is represented by a straight line parallel to the x-axis (a horizontal line) as depicted in figure 4.2.7.



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For a vertical line (that's a line parallel to the y-axis), however, the slope does not exist. The reason being, considering the formula of the gradient,

$$g = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_0 - y_1}{x_0 - x_0} = \frac{y_0 - y_1}{0}$$

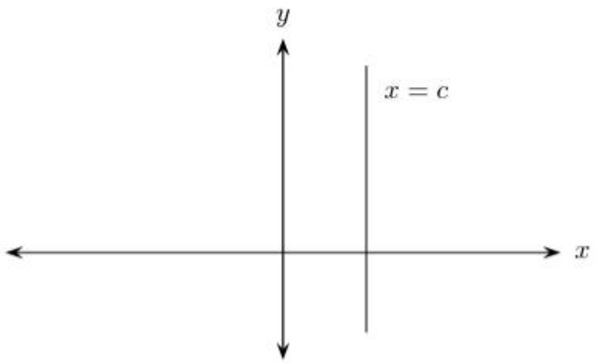
Which is undefined because x does not change.

- **A straight line parallel to the y-axis.**

  In this case we would have $x_2 = x_1$ and

$$\frac{y_2 - y_1}{0}.$$

Division by zero is not defined. We say that the slope becomes **infinite** in this case. The line is **vertical**, that is, as shown in figure 4.2.8.



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When there are two or more lines such that the slopes are the same, for example,

$$y = gx - 13$$

or

$$y = gx + 2$$

then the lines are parallel to one another.

The straight line is just one of many well known functions in mathematics. Which brings us to functions. But before that we define a few concepts.

A dependent variable is a variable whose value depends on the values of other variables of the formula or equation (in other words the subject of the formula is the dependent variable.) An independent variable, on the other hand, is the variable on which the dependent variable depends. There can be many independent variables but only one dependent variable at a time.

A function is therefore, a relation between two or among three (or more variables) that associates every set of the values (single for each variable) of the independent variables with exactly one value of the dependent variable. So this means for an (x, y) function for every value of x you have exactly one y.

So

$$f(x)$$

means the function f depends x.

$$f(x, y, z)$$

tells us that the function f depends on the variables x, y and Z. The $f(x, y, z)$ represents the dependent variable. Other functions we have to consider include the parabolas, exponential functions and the log functions.

A parabola has the formula

$$f(x) = ax^2 + bx + c.$$

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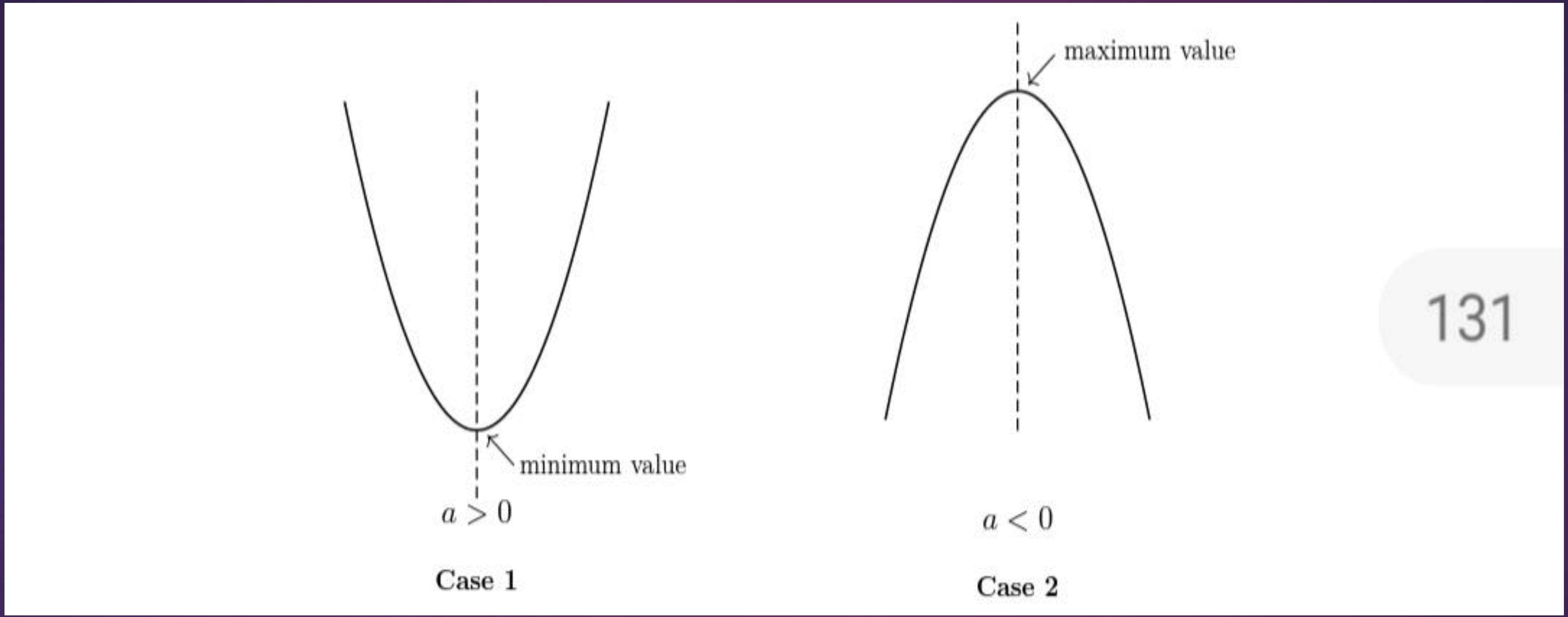
a and b are coefficients and c is a constant. The parabola also has the shapes



depending on the sign of the coefficient of x^2.

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The formula

$$x = -\frac{b}{2a}$$

is the x value at the turning point (vertex) of the parabola. If the $a < 0$ then the point $(-\frac{b}{2a}, -\frac{b^2}{4a} + c)$ is our maximum point (the highest point of the function.) But suppose $a > 0$ consequently $(-\frac{b}{2a}, -\frac{b^2}{4a} + c)$ is the minimum point (the lowest point of the function.)

At

$$x = -\frac{b}{2a}$$

y will be

$$y = -\frac{b^2}{4a} + c.$$

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Example 4E

Find the turning points of:

1. $f(x) = 2x^2 - x - 3$
2. $f(x) = 4x^2 - 16x + 16$
3. $f(x) = -3x^2 + 3x - 2$

Solution 1

$$x = -\frac{b}{2a} = -\frac{-1}{2(2)} = \frac{1}{4}$$

$$y = -\frac{b^2}{4a} + c = -\frac{(-1)^2}{4(2)} + (-3) = -\frac{1}{8} - 3 = -3\frac{1}{8}.$$

Therefore, the vertex is at $(\frac{1}{4}; -3\frac{1}{8})$. And $a = 2 > 0$ so the turning point is the minimum so the graph is facing up.

Solution 2

$$x = -\frac{b}{2a} = -\frac{-16}{2(4)} = 2$$
$$y = -\frac{b^2}{4a} + c = -\frac{(-16)^2}{4(4)} + 16 = 0.$$

The turning point is $(2, 0)$. The graph is facing up (opening upward.)

Solution 3

$$x = -\frac{b}{2a} = -\frac{3}{2(-3)} = -\frac{3}{-6} = \frac{1}{2}$$
$$y = -\frac{b^2}{4a} = -\frac{(3)^2}{4(-3)} = -\frac{9}{-12} = \frac{3}{4}.$$

The turning point (maximum point) is $\left(\frac{1}{2}, \frac{3}{4}\right)$. $-3 < 0$ so the parabola is opening down-ward.

We can also get the intercepts of both x and y.

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The y-intercept is where x is 0.

$$y = a\,(0)^2 + b(0) + c = c$$

The x-intercept is where y is 0. That is where

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 4F

From Example 4E work out the x and y intercepts of each parabola.

Solution 1

$$\boldsymbol{y = c = -3}$$

$$\boldsymbol{x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)} = \frac{1 \pm \sqrt{25}}{4}}$$

$$x = \frac{3}{2}$$

And

$$x = -1.$$

Solution 2

$$y = c = 16$$

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$= \boldsymbol{\frac{-(-16) \pm \sqrt{(-16)^2 - 4(4)(16)}}{2(4)}}$$

$$= \boldsymbol{\frac{16}{8} = 2}$$

This means there is only one x-intercept at $x = 2$ since $b^2 - 4ac$ is equal to zero. This means only the turning point touches the x-axis.

Solution 3

$$y = c = -2$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{3^2 - 4(-3)(-2)}}{2(3)}$$

$$x = \frac{-3 \pm \sqrt{-15}}{6}$$

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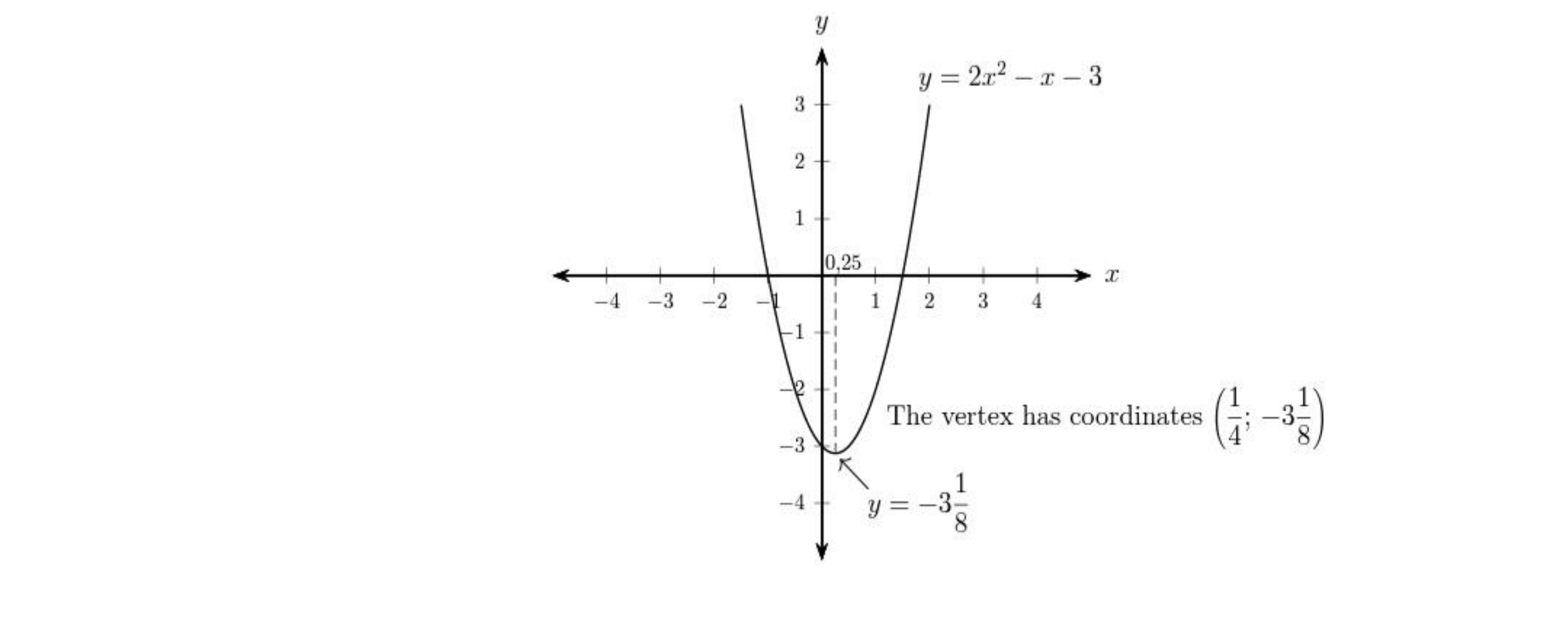
For $b^2 - 4ac < 0$ there is no real number so a real number does not exist. And $-15 < 0$. The parabola does not touch the x-axis.

Example 4G

The graphs of $1, 2$ and 3.

1.



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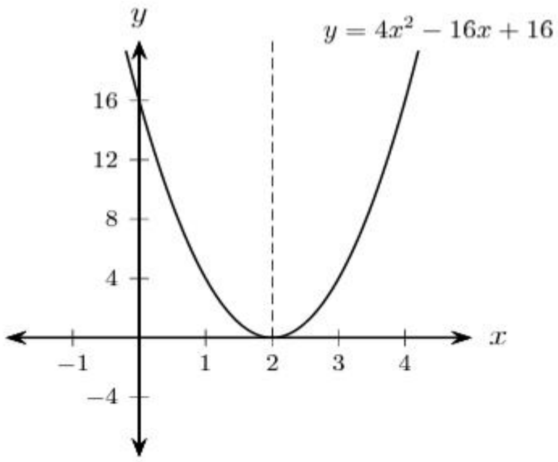
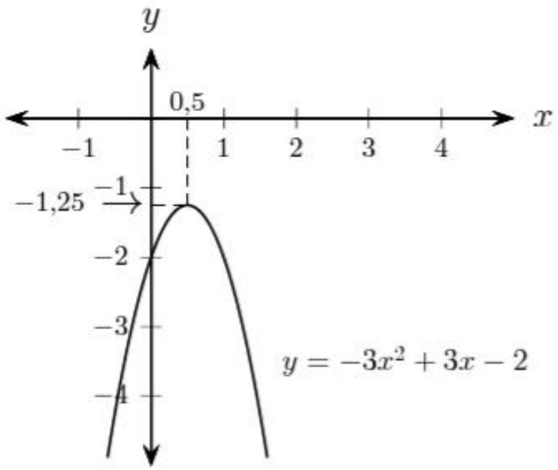
2.



3.



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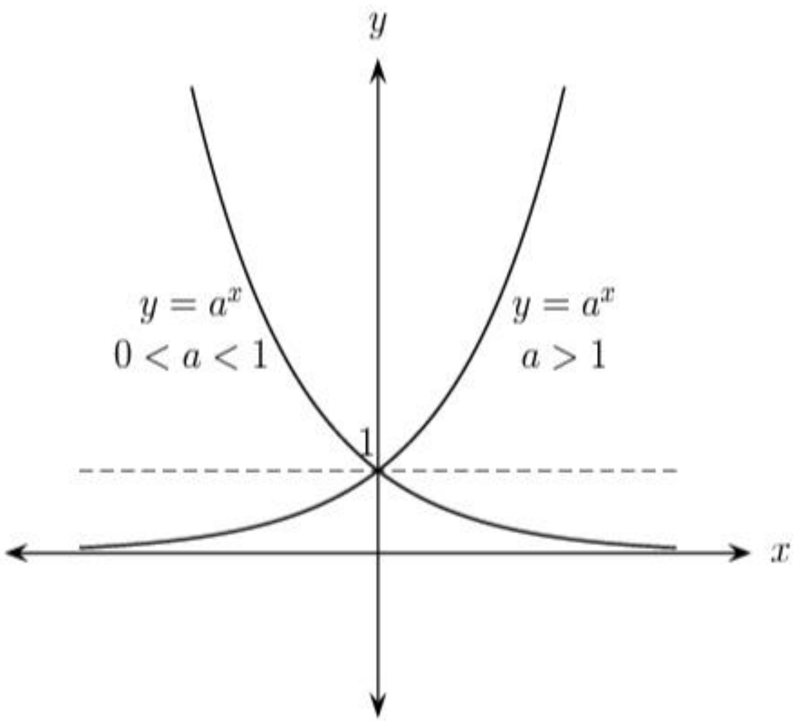
The shapes of the exponential graphs are something like this



The equation of the exponential graph is

$$f(x) = a^x.$$

Where a is a constant. Notice that if the a was just 1 then the function would just be

$$y = f(x) = 1^x = 1$$

a horizontal line.

Now the $\log$ function is the inverse of the exponential function where a is the base of the $\log$ and x is the argument or the independent variable. Written as

$$f(x) = \log_a x\,.$$

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Exercises 4A

1. (a) Determine the equation of the straight line through the points (1; 2) and (3; 3).
   (b) Find the intercepts on the x- and the y-axe of the line in (a).
   (c) Is the line in (a) parallel to the line

$$y = 2 + x?$$

   Why or why not?

   (d) Draw the lines of (a) and (c) on one graph.

2. Consider the lines

$$y = 5 + 2x$$

   and

$$y = 2 + x.$$

   What are their intercepts on the axes? Are they parallel or not? What is the vertical distance between the lines at

$$x = 3{,}5?$$

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3. Draw the following lines on one graph:

   (a) $x = 2$

   (b) $y = 4x$

   (c) $y = -2x - 3$

4. A bus agency has room for 60 people on a bus tour. If they charge R6 000 per person, they will be able to fill the bus. They know from experience that if they increase the price of the tour by R500 they will lose three customers. Determine the price function if the price p (in rand) is a linear function of the demand (number of customers).

Exercises 4B

1. Determine the intercepts on the axes and the vertices of each of the following quadratic functions and sketch the curves:

   (a) $y = -0{,}4x^2 + 0{,}2x + 1{,}2$

   (b) $y = -x^2 - 2x - 1$

   (c) $y = x^2 + 4x + 5$

   (d) $y = -x^2 + 9$

2. If

$$d = p^2 - 45p + 520$$

describes a weekly demand for a certain ice cream in litres, with p the price per litre and d the demand, what is the price per litre that minimises the weekly demand? What is the minimum weekly demand?

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Chapter Five

LINEAR SYSTEMS

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Solving a linear system of equations in one independent variable means finding where the equations meat or intercept. This is where,

$$f(x) = g(x),$$

the equations are equal. From which we manipulate using division, multiplication, subtraction and addition to get the value of the independent variable.

Example 5A

Suppose we have the system of equations

$$2x + 5y = 13$$

and

$$3x + 4y = 9.$$

For the first equation

$$f(x) = y = \frac{13}{5} - \frac{2}{5}x.$$

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Then the second equation will be

$$g(x) = y = \frac{9}{4} - \frac{3}{4}x.$$

Then we equate the functions $g(x) \ = \ f(x)$. This brings us to

$$\frac{9}{4} - \frac{3}{4}x = \frac{13}{5} - \frac{2}{5}x.$$

The

$$LCM(4, 5) \ = \ 20.$$

This means we can

$$20\left(\frac{9}{4} - \frac{3}{4}x\right) = 20\left(\frac{13}{5} - \frac{2}{5}x\right)$$

$$45 \ - 15x = 52 \ - 8x$$

(By the Distributive Law of Multiplication over Subtraction)

$$-8x + 15x = 45 \ - 52$$

$$7x = -7$$

$$x = -\frac{7}{7} = -1 \ldots (1)$$

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We can now substitute …(1) into either one of our two equations. Choosing

$$3x + 4y = 9$$

we have

$$3(-1) + 4y = 9.$$
$$-3 + 4y = 9$$
$$4y = 9 + 3$$
$$4y = 12$$
$$y = \frac{12}{4}$$
$$y = 3.$$

Inequalities use the symbols $<, >, \leq$ and $\geq$ instead of $=$. That is smaller than, greater than, smaller than or equal and greater than or equal, respectively. A system of linear inequalities in one variable is where more than one inequality is used to find and shade a specific region above or below each linear equation.

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Example 5B

Find the possible values of x in the following:

1. $5x - 15 < 0$
2. $5x + 6 \geq 6x - 5$
3. $11 \geq 6 - 4x$
4. $4x + 4 < 1{,}5x - 6$

Solution 1

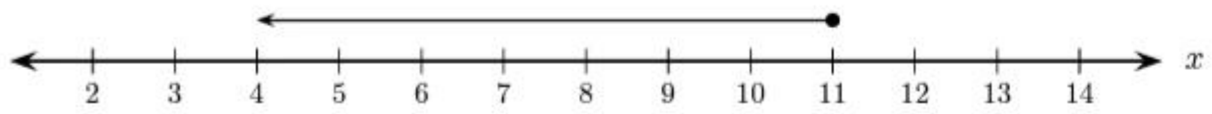
$$\mathbf{5x - 15 < 0}$$

$$(5x - 15) + 15 < 0 + 15$$
$$5x < 15$$
$$x < \frac{15}{5} = 3$$
$$x < 3.$$

Solution 2

$$5x + 6 \geq 6x - 5$$
$$(5x + 6) - 6 \geq (6x - 5) - 6$$
$$5x \geq 6x - 11$$
$$5x - 6x \geq 6x - 11 - 6x$$
$$-1x \geq -11$$
$$x \leq -\frac{11}{-1} = 11$$

The solution consists of all real values less or equal to 11. This is indicated on the line below:



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Solution 3

$$11-6\ \geq 6\ -4x\ -6$$
$$5 \geq -4x$$
$$-\frac{5}{4} \leq x$$

Solution 4

$$\mathbf{4x\ +4\ -4 < 1,5x\ -6\ -4}$$
$$\mathbf{4x < 1,5x\ -10}$$
$$\mathbf{4x\ -1,5x < 1,5x\ -10\ -1,5x}$$
$$\mathbf{2,5x < -10}$$
$$\mathbf{x < -\frac{10}{2,5}}$$
$$\mathbf{x < -4}$$

When it is a system of inequalities it means we are working with more than one inequality. But to first indicate how the inequality sketches look. We have Example 5C.

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Example 5C

Plot $-x + 2y - 2 \geq 0$.

The first thing we do is get y to be the subject of the formula by rearranging our formula.

$$(-x + 2y - 2) + x + 2 \geq x + 2$$
$$2y \geq x + 2$$
$$y \geq \frac{1}{2}x + \frac{2}{2} = \frac{1}{2}x + 1$$

To plot we first need the line

$$-x + 2y - 2 = 0$$
$$2y = x + 2$$
$$y = \frac{1}{2}x + \frac{2}{2}$$
$$y = \frac{1}{2}x + 1.$$

Our region will therefore be the part above the line including the line itself since the values of y are those greater than or equal to our line $\frac{1}{2}x + 1$ for every x.

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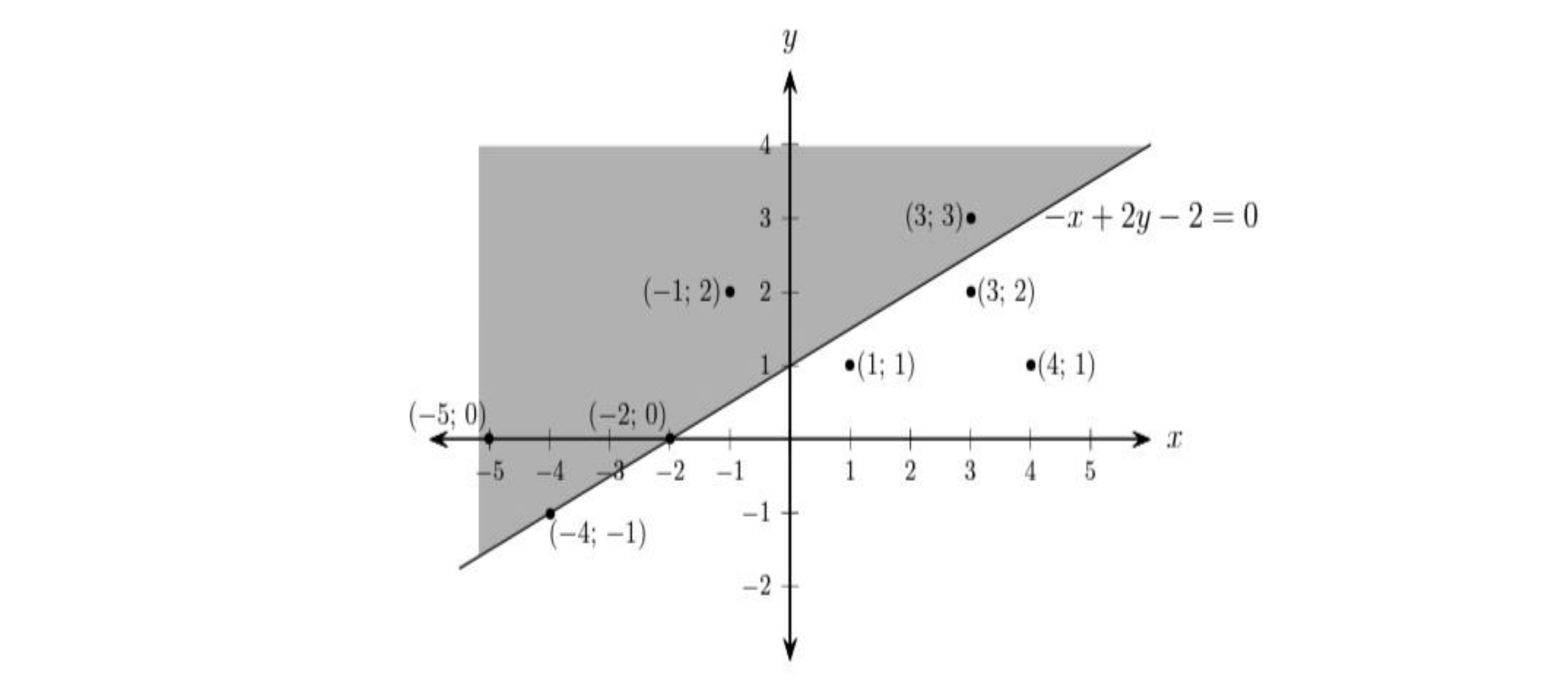

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Example 5D

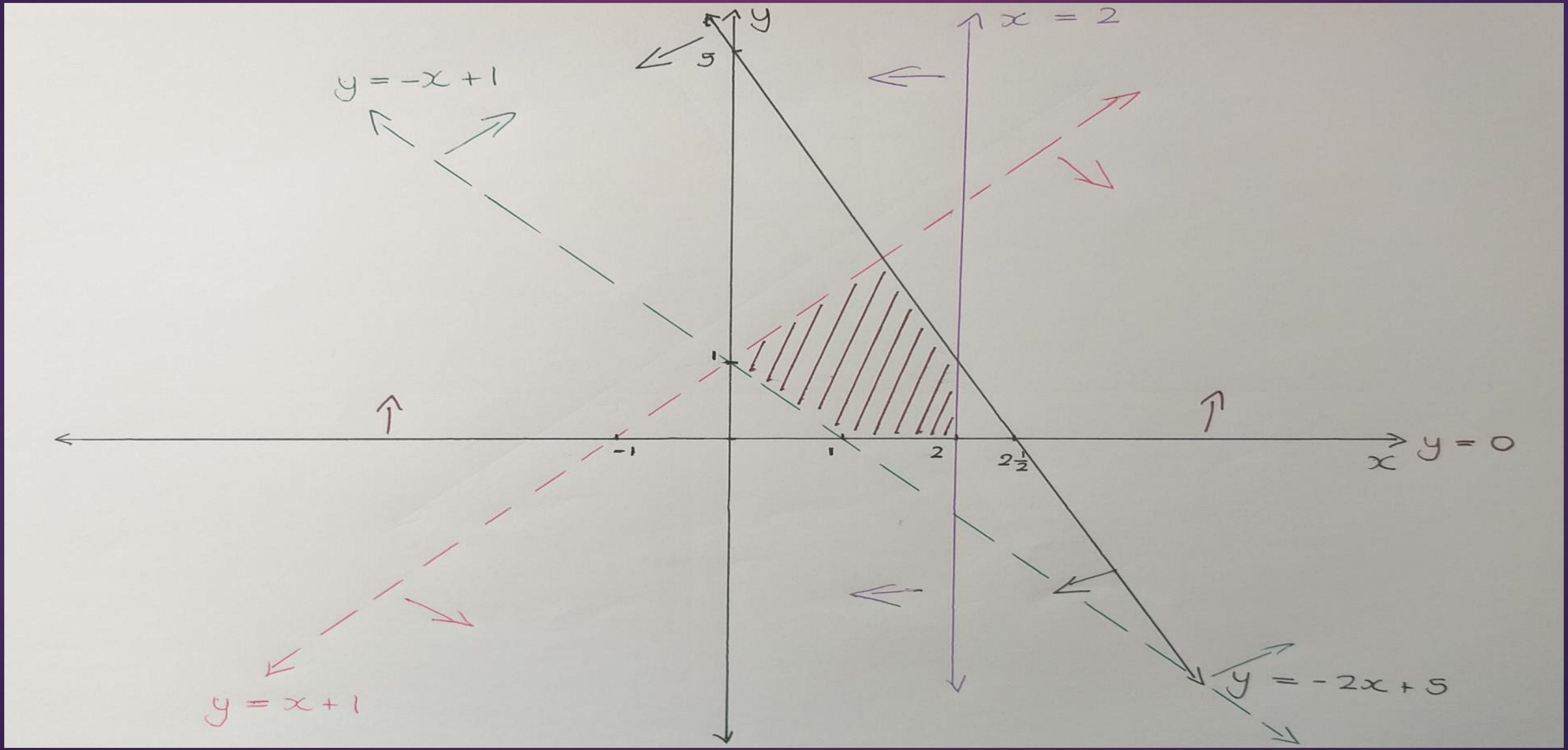
For a System of five inequalities consider the following inequalities

$$2x + y\ - 5 \leq 0 \ldots (1)$$
$$x\ - 2 \leq 0 \ldots (2)$$
$$-x + y\ - 1 \leq 0 \ldots (3)$$
$$x + y\ - 1 > 0 \ldots (4)$$
$$y \geq 0 \ldots (5)$$

The corresponding equations of straight lines will result as

$$(2x + y\ - 5) - 2x + 5 \leq 0\ - 2x + 5$$
$$y \leq -2x + 5 \ldots (\mathbf{I1})$$
$$y = -2x + 5 \ldots (\mathbf{E1})$$
$$(x\ - 2) + 2 \leq 0 + 2$$
$$x \leq 2 \ldots (\mathrm{I}2)$$
$$x = 2 \ldots (\mathrm{E}2)$$
$$(-x + y\ - 1\,) + x + 1 < 0 + x + 1$$
$$y < x + 1 \ldots (\mathrm{I}3)$$
$$y = x + 1 \ldots (\mathrm{E}3)$$

$$(x + y - 1) - x + 1 > 0 - x + 1$$
$$y > -x + 1 \ldots (\mathbf{I4})$$
$$\boldsymbol{y = -x + 1 \ldots (E4)}$$
$$y \geq 0 \ldots (\mathbf{I5})$$
$$\boldsymbol{y = 0 \ldots (E5)}$$



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Chapter Six

APPLICATIONS OF DIFFERENTIATION

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To kick off this this chapter I define differentiation. The derivative of polynomials and general rules about differentiation.

Let

$$f(x) = x^n.$$

The derivative of $f(x)$, which is denoted $f'(x)$ or $\frac{d}{dx}f(x)$, will be

$$f'(x) = \frac{d}{dx}f(x) = \; nx^{n-1}.$$

Then consider the functions $h(x)$ and $g(x)$.

Suppose

$$t(x) = h(x) + g(x).$$

Then the derivative which is the slope at every possible x value will be

$$t'(x) = \frac{d}{dx}t(x) = h'(x) + g'(x),$$

the sum of the independent derivatives of $h(x)$ and $g(x)$.

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If $p(x) = y = ah(x)$ where a is a constant then

$$p'(x) = \frac{d}{dx}p(x) = \frac{dy}{dx} = ah'(x)$$

But if $q(x) = b + g(x)$ where b is a constant then

$$q'(x) = 0 + g'(x),$$

because the derivative of any constant is zero. Derivative means differentiation.

If we have a profit function $P(x)$ then the marginal profit function will be the derivative of the profit function, $P'(x)$ at x. Furthermore, when we have a cost function, $C(x)$, then the marginal cost function with be the derivative of the cost function, $C'(x)$ at x. To find the x value at the maximum profit we equate the marginal profit, $P'(x)$, to zero. That's

$$P'(x) = 0.$$

To find the x value at the minimum cost we equate the marginal cost function to zero. That is,

$$C'(x) = 0.$$

Example 6A

Let

$$R(x) = 10x - \frac{1}{1000}x^2$$

be our revenue function. Suppose our cost function is

$$C(x) = 7000 + 2x.$$

Now calculate the maximum profit.

Profit is revenue minus cost. And we first find the x value at the maximum profit.

$$P(x) = R(x) - C(x)$$

$$P(x) = \left[10x - \frac{1}{1000}x^2\right] - [7000 + 2x]$$

$$P(x) = 8x - \frac{1}{1000}x^2 - 7000$$

$$P(x) = 8(1x^{1-1}) - \frac{1}{1000}(2x^{2-1}) - 0 = 8\mathrm{x}^0 - \frac{2}{1000}\mathrm{x} = 0$$

$$8 - \frac{1}{500}x = 0$$

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$$\frac{1}{-500}x = -8$$
$$x = -500\,(-8) = 4000$$

Since we have the x value we can now plug it into the profit function.

$$P(4000) = 8(4000)\; - \frac{1}{1000}(4000)^2 - 7000$$
$$= 32000\; - \frac{16\,000\,000}{1000}\; - 7000$$
$$= 32000\; - 16\,000\; -\; 7000$$
$$= 9000$$

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Chapter Seven

MATHEMATICS OF FINANCE

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Interest is measured in proportions or percentages. It is a fraction of the original amount (Principle.)

A simple interest of 12% per period (per annum in the case of years) for five years gives us a total interest amount of

$$Pit = (0{,}12) \times 5 \times P = 0.6P$$

P is the original or invested amount. i and t are the interest rate and time (in years), respectively. If we add this to the original amount is

$$P + Pit = P(1 + it) = S$$

This is the sum you receive at the end of the investment period. Simple interest is calculated on the original amount every period which differentiates it from compound interest. Then

$$S = P(1 + it)$$

is the formula for finding the sum when P is invested at a simple interest.

The defining characteristic of compound interest is that the interest itself earns interest.

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So for the first period (year)

$$S = P(1+i).$$

From year two we reinvest the last value or amount of the investment that would be

$$S = P(1+i)$$

but the principle in this case is the amount we got back from the first year's investment. Meaning

$$S = [P(1+i)](1+i) = P(1+i)^2$$

If we continue this way we end up with the formula for valuing an investment amount at a compound interest.

$$S = P(1+i)^t$$

When it is said that interest is compounded monthly, every three months or biannually it means our period is now monthly, every quarter or every six months and so on.

This means we divide our interest rate by the number of compounds per year, so for monthly we will divide the interest rate by 12 months and for biannually we divide the interest by two because that's every six months there are two of those in a year.

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Example 7A

Calculate the simple interest and sum accumulated when $R5000$ is borrowed for 90 days at a simple interest of 12% per annum. Note that the time here is a fraction of a year.

The formula

$$S = P + Pit \ = P + I$$

$$I = R5000 \times \frac{12}{100} \times \frac{90}{365} = R\ 147,95$$

$$S = R5000 + R147,95 = R5147,95$$

Example 7B

You invest $R1000$ at a simple interest rate of 10% per annum for four years. What is the total interest that you receive?

$$I = Pit = R1000 \times \frac{10}{100} \times 4 = R400$$

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Example 7C

What is the total amount available in 2,5 years if $R2000$ is invested at an interest rate of 12% per annum compounded quarterly (every three months.)

$$S = P\left(1 + \frac{i}{4}\right)^{4\times t}$$

By dividing the interest rate by four we get a quarterly effective interest rate. This means we have to convert our time intervals to quarters so 2,5 years is $2{,}5 \times 4 = 10$ quarters.

$$S = R2000 \times \left(1 + \frac{0.12}{4}\right)^{10}$$

$$S = R2000 \times 1{,}343916\ldots$$

$$S = R2687{,}83$$

I now endeavor to apply logarithms in the next example.

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Example 7D

If you invest $R10000$ in an account that pays 15% interest per annum compounded monthly. How many years will it take your investment to double?

We divide the interest rate by the numbers of compounding per year.

$$i = \frac{0.15}{12} = 0.0125 \text{ (Monthly)}$$

$$S = 2P = 2 \times R10000 = R20\,000 \text{ (Double the Principle)}$$
$$P = R10\,000$$

$$S = P(1+i)^{12t} \text{ (} t \text{ is in years so } 12t \text{ is months)}$$
$$20000 = 10000 \times (1+0.0125)^{12t}$$

I want t and it is in the power of the accumulation factor.

$$\frac{20000}{10000} = (1{,}0125)^{12t}$$
$$2 = (1{,}0125)^{12t}$$

We use the $\log$ property that drops the power. To achieve this we $\log$ both sides.

$$\log 2 = \log(1{,}0125)^{12t}$$
$$12t \times \log(1{,}0125) = \log 2$$
$$12t = \frac{\log 2}{\log(1{,}0125)}$$
$$t = \frac{\log 2}{12 \times \log(1{,}0125)}$$
$$t = 4{,}6498 \text{ years}$$

Introducing annuities, an annuity is a series of payments just like monthly payments on a loan. For annuities we can calculate the present value and the accumulated value i.e. the sum.

The Present Value equation is

$$P = R\left[\frac{(1+i)^t - 1}{i(1+i)^t}\right].$$

The Future Value or Sum Accumulated will then be

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$$S = R\left[\frac{(1+i)^t - 1}{i}\right].$$

The R represents the regular payments. As an example, consider the following.

Example 7E

Brenda secured a 20-year loan for $R400\ 000$. She repays the loan in equal monthly payments or instalments. The annual interest rate 16%, compounded monthly. What's Brenda's minimum monthly payments.

We use the present value formula of annuities to get R because the $R400\ 000$ is the present value.

$$P = R\left[\frac{(1+i)^{12t} - 1}{i(1+i)}\right]$$

Changing the subject of the formula

$$R = P\left[\frac{(1+i)^{12t} - 1}{i(1+i)^{12t}}\right]^{-1} = \frac{Pi(1+i)^{12t}}{(1+i)^{12t} - 1} = \frac{400000 \times \frac{0,16}{12} \times \left(1 + \frac{0,16}{12}\right)^{12\times 20}}{\left(1 + \frac{0,16}{12}\right)^{12\times 20} - 1}$$

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$$R = R\ 5\ 565{,}02$$

This was because our

$$i = \frac{0{,}16}{12}$$

and

$$t = 20$$

together with the Present Value of $R400\ 000$.

A question that uses the future value formula comes up next.

Example 7F

Samantha decides to invest money regularly to become a millionaire in ten years' time. She manages to secure an interest rate of twenty four percent per annum, compounded monthly. She starts making monthly payments into this account. How much would she need to invest every month in order to have a sum of One Million Rands at the end of ten years?

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The One Million Rands is a future value so we will use the sum formula for annuities.

$$S = R\left[\frac{(1+i)^{12t}-1}{i}\right]$$

We need R so changing the subject of the formula we arrive at

$$R = \frac{Si}{(1+i)^{12t}-1}$$

$$S = R1\ 000\ 000$$

$$i = \frac{0{,}24}{12} = 0{,}02$$

$$t = 10$$

$$R = \frac{1\ 000\ 000 \times 0{,}02}{(1+0{,}02)^{12\times 10} - 1} = R2048{,}10$$

Payments to a loan go toward the interest payment and the capital portion of the loan. Amortisation is when a loan is paid off with this series of instalments. The loan is said to be amortised.

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Example 7G

Refer back to Example 7E on Brenda's loan and consider this armotisation question.

What will the outstanding balance be after all the minimum payments have been made for fifteen years? Assume the interest remained fixed for the whole period.

From the first payment the amount remaining will be

$$A_R = 400000(1+i) - R$$

Second payment leaves

$$\begin{aligned} A_R &= [400000(1+i) - R](1+i) - R] \\ &= 400000(1+i)^2 - R(1+i) - R \end{aligned}$$

The third one results in

$$\begin{aligned} A_R &= [400000(1+i)^2 - R(1+i) - R](1+i) - R] \\ &= 400000(1+i)^3 - R(1+i)^2 - R(1+i) - R \\ &= 400000(1+i)^3 - [R(1+i)^2 + R(1+i) + R] \end{aligned}$$

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The last term of the equation is the future value of an annuity with payments R and the monthly interest rate i. This is as a result of the fifteen year formula

$$A_R = 400000(1+i)^{12t} - [R(1+i)^{12t-1} + R(1+i)^{12t-2} + \cdots + R]$$

$$t = 15 \text{ years}$$

$$i = \frac{0,16}{12}$$

$$R = 5565,0238 \text{ (From Example 7E)}$$

Therefore,

$$A_R = P(1+i)^{12t} - R\left[\frac{(1+i)^{12t} - 1}{i}\right]$$

$$A_R = P(1+i)^{12t} - Ra_{\overline{12t}|i}$$

This is all you have to know. Just the formula and how to use it to amortise loans.

The amount remaining is then

$$A_R = 400000\left(1+\frac{0,16}{12}\right)^{12\times 15} - 5565,0238 \times \left[\frac{\left(1+\frac{0,16}{12}\right)^{12\times 15} - 1}{\frac{0,16}{12}}\right] = R228\,843,24$$

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Simple discount is much like simple interest in idea. The discount is calculated on the principle for each time interval going all the way to the maturity or expiry date. So for the sum S and principle P a simple discount of d over a period of t years looks a little something like

$$P = S - Sd - Sd - Sd - \cdots$$
$$P = S - (Sd + Sd + \cdots)$$
$$P = S - Sdt$$
$$P = S(1 - dt).$$

Compound discount is also similar to compound interest. This means for each interval the discount is calculated on the last balance, that is to say, right at the beginning of the new period. For this reason, the principle P when applying a discount d on a sum S for t years will amount to

$$P = [[[S(1-d)](1-d)](1-d)] \ldots$$
$$P = S(1-d)^t$$

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Example 7H

A loan that has to be repaid in 9 months time has a discount rate of 12% per annum. The present value of which is $R8000$.

a) What is the actual amount of money you receive now once the discount has been applied or deducted from the future value?

ANSWER

This is $R8000$. Because the loan payment at the end of 9 months is $R8000 \times (1 - dt)^{-1}$, the amount we discount.

b) What is the future value in order to actually receive the present value of R8000?

ANSWER

$$P = S(1 - dt)$$
$$S = P(1 - dt)^{-1}$$
$$S = R8000\left(1 - 0.12\frac{9}{12}\right)^{-1} = R8791,21$$

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c) Determine the equivalent simple interest rate of the loan.

ANSWER

We know that our interest amounts to

$$Pit = R8791{,}21 - R8000$$

$$R8000 \times \frac{9}{12} \times i = 791{,}21$$

$$i = \frac{12 \times 791{,}21}{9 \times 8000}$$

$$i = 0{,}131868 = 13{,}19\%$$

Or

$$(1 \ - dt)^{-1} = (1 + it)$$

$$\left(1 - 0{,}12\frac{9}{12}\right)^{-1} = \left(1 + \frac{9}{12}i\right)$$

$$\frac{9}{12}i = (1 \ - 0{,}09)^{-1} - 1$$

$$i = \frac{12[0{,}91^{-1} - 1]}{9} = 0{,}131868 \ldots = 13{,}19\%$$

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