General Relativity II

Einstein Equations: History

- A nice description of the whole story is in the Thorne's book.
- Einstein made numerous mistakes in deriving his equations, making 4 presentations at the Berlin Academy, each time saying "sorry, I made a mistake in the last week presentation" and no one threw rotten tomatoes at him.
- Mathematician Hilbert derived Einstein equations a week before Einstein but refused to claim the credit.

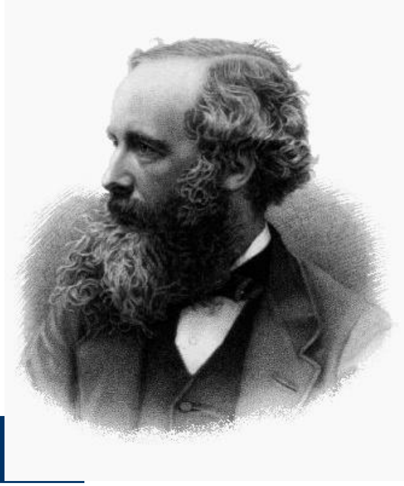
Einstein Equations: Physics

- Equations of GR relate the ***curvature*** of space-time and its contents (***matter + energy***).
- Symbolically, they are

Curvature = Matter + Energy

- Which side is the cause?
  - **A**: curvature
  - **B**: matter + energy

Physics of 1887

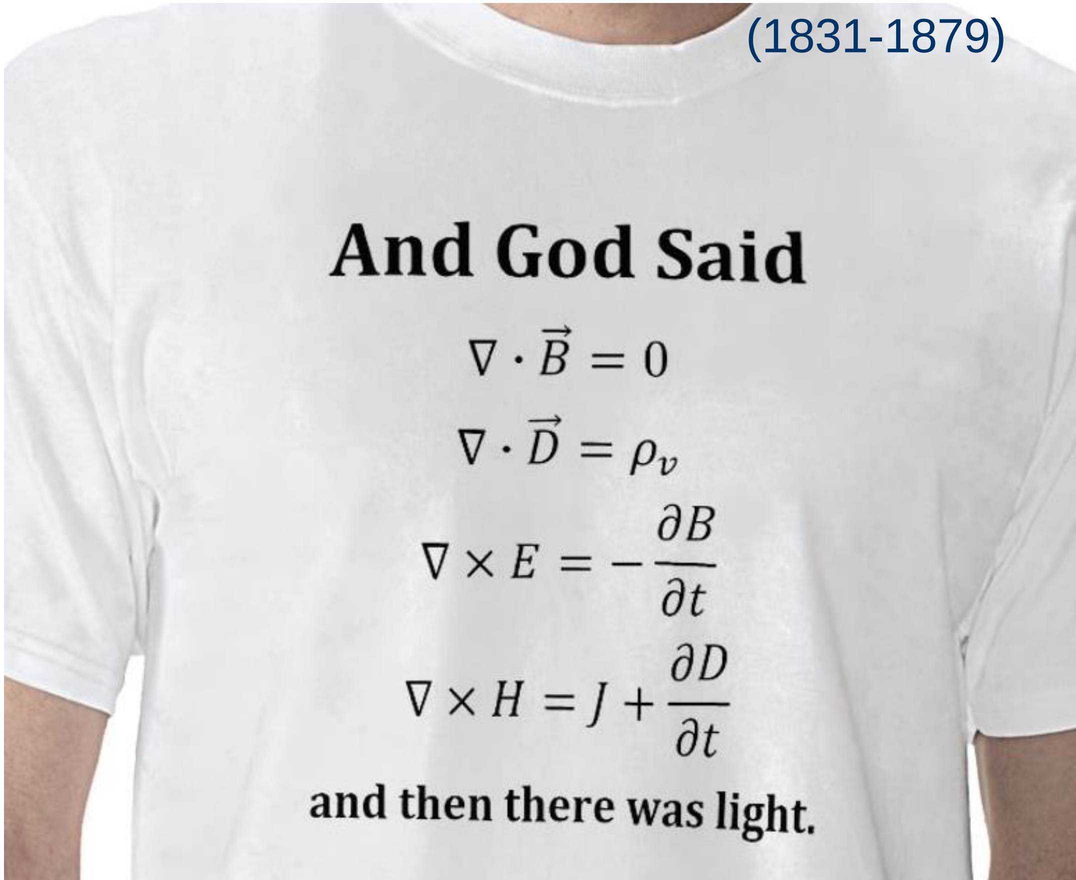


Isaac Newton
(1642-1727)

James Maxwell
(1831-1879)

$$\vec{F} = m\vec{a}$$

$$F_g = G\frac{mM}{R^2}$$



Einstein Equations: Math

$$\frac{1}{2}g^{rs}\left(-\frac{\partial^2 g_{ij}}{\partial x^r \partial x^s}+\frac{\partial^2 g_{is}}{\partial x^r \partial x^j}+\frac{\partial^2 g_{rj}}{\partial x^i \partial x^s}-\frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right)+\frac{1}{4}g^{qp}\left(-\frac{\partial g_{is}}{\partial x^p}+\frac{\partial g_{pi}}{\partial x^s}+\right.$$

$$\left.\frac{\partial g_{ps}}{\partial x^i}\right)\times\left(\frac{\partial g_{qj}}{\partial x^r}+\frac{\partial g_{qr}}{\partial x^j}-\frac{\partial g_{rj}}{\partial x^q}\right)-\frac{1}{4}g^{qp}\left(-\frac{\partial g_{ij}}{\partial x^p}+\frac{\partial g_{pi}}{\partial x^j}+\frac{\partial g_{pj}}{\partial x^i}\right)\left(\frac{\partial g_{qr}}{\partial x^s}+\right.$$

$$\left.\frac{\partial g_{qs}}{\partial x^r}-\frac{\partial g_{rs}}{\partial x^q}\right)-\frac{1}{4}g_{ij}g^{rs}g^{uv}\left(-\frac{\partial^2 g_{rs}}{\partial x^u \partial x^v}+\frac{\partial^2 g_{rv}}{\partial x^u \partial x^s}+\frac{\partial^2 g_{us}}{\partial x^r \partial x^v}-\frac{\partial^2 g_{uv}}{\partial x^r \partial x^s}\right)+$$

$$\frac{1}{8}g_{ij}g^{rs}g^{uv}g^{qp}\left(\frac{\partial g_{qr}}{\partial x^v}+\frac{\partial g_{qv}}{\partial x^r}-\frac{\partial g_{rv}}{\partial x^q}\right)\left(\frac{\partial g_{ps}}{\partial x^u}+\frac{\partial g_{pu}}{\partial x^s}-\frac{\partial g_{us}}{\partial x^p}\right)-$$

$$\frac{1}{8}g_{ij}g^{rs}g^{uv}g^{qp}\left(\frac{\partial g_{qr}}{\partial x^s}+\frac{\partial g_{qs}}{\partial x^r}-\frac{\partial g_{rs}}{\partial x^q}\right)\left(\frac{\partial g_{pu}}{\partial x^v}+\frac{\partial g_{pv}}{\partial x^u}-\frac{\partial g_{uv}}{\partial x^p}\right)=\frac{8\pi G}{c^4}T_{ij}.$$

Modern Particle Physics

1

$$-\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu +$$
$$\tfrac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu -$$

2

$$M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \tfrac{1}{2}\partial_\mu H \partial_\mu H -$$
$$\tfrac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2\phi^+\phi^- - \tfrac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \tfrac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\tfrac{2M^2}{g^2} +$$
$$\tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \tfrac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu -$$
$$W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu -$$
$$W^-_\nu\partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu -$$
$$W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)] - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu +$$
$$\tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) +$$
$$g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu -$$
$$W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] -$$
$$\tfrac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2] -$$
$$gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \tfrac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) -$$
$$W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \tfrac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ -$$
$$\phi^+\partial_\mu H)] + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) +$$
$$igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) +$$
$$igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \tfrac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] -$$
$$\tfrac{1}{4}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-] - \tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \tfrac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- -$$
$$g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j -$$

$$g^1 s_w^2 A_\mu A_\mu\phi^+\phi^- - \bar{e}^\lambda(\gamma\partial + m_e^\lambda)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m_u^\lambda)u^\lambda_j -$$

3

$$\bar{d}^\lambda_j(\gamma\partial + m_d^\lambda)d^\lambda_j + igs_w A_\mu[-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)] +$$
$$\tfrac{ig}{4c_w}Z^0_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 -$$
$$1 - \gamma^5)u^\lambda_j) + (\bar{d}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 - \gamma^5)d^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}W^+_\mu[(\bar{\nu}^\lambda\gamma^\mu(1 + \gamma^5)e^\lambda) +$$
$$(\bar{u}^\lambda_j\gamma^\mu(1 + \gamma^5)C_{\lambda\kappa}d^\kappa_j)] + \tfrac{ig}{2\sqrt{2}}W^-_\mu[(\bar{e}^\lambda\gamma^\mu(1 + \gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\lambda\kappa}\gamma^\mu(1 +$$
$$\gamma^5)u^\lambda_j)] + \tfrac{ig}{2\sqrt{2}}\tfrac{m_e^\lambda}{M}[-\phi^+(\bar{\nu}^\lambda(1 - \gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1 + \gamma^5)\nu^\lambda)] -$$

4

$$\tfrac{g}{2}\tfrac{m_e^\lambda}{M}[H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda)] + \tfrac{ig}{2M\sqrt{2}}\phi^+[-m_d^\kappa(\bar{u}^\lambda_j C_{\lambda\kappa}(1 - \gamma^5)d^\kappa_j) +$$
$$m_u^\lambda(\bar{u}^\lambda_j C_{\lambda\kappa}(1 + \gamma^5)d^\kappa_j] + \tfrac{ig}{2M\sqrt{2}}\phi^-[m_d^\lambda(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 + \gamma^5)u^\kappa_j) - m_u^\kappa(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1 -$$
$$\gamma^5)u^\kappa_j] - \tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) -$$
$$\tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 -$$

5

$$\tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_w W^+_\mu(\partial_\mu\bar{X}^0 X^- - \partial_\mu\bar{X}^+ X^0) + igs_w W^+_\mu(\partial_\mu\bar{Y}X^- -$$
$$\partial_\mu\bar{X}^+ Y) + igc_w W^-_\mu(\partial_\mu\bar{X}^- X^0 - \partial_\mu\bar{X}^0 X^+) + igs_w W^-_\mu(\partial_\mu\bar{X}^- Y -$$
$$\partial_\mu\bar{Y}X^+) + igc_w Z^0_\mu(\partial_\mu\bar{X}^+ X^+ - \partial_\mu\bar{X}^- X^-) + igs_w A_\mu(\partial_\mu\bar{X}^+ X^+ -$$
$$\partial_\mu\bar{X}^- X^-) - \tfrac{1}{2}gM[\bar{X}^+ X^+ H + \bar{X}^- X^- H + \tfrac{1}{c_w^2}\bar{X}^0 X^0 H] +$$
$$\tfrac{1-2c_w^2}{2c_w}igM[\bar{X}^+ X^0\phi^+ - \bar{X}^- X^0\phi^-] + \tfrac{1}{2c_w}igM[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] +$$
$$igMs_w[\bar{X}^0 X^-\phi^+ - \bar{X}^0 X^+\phi^-] + \tfrac{1}{2}igM[\bar{X}^+ X^+\phi^0 - \bar{X}^- X^-\phi^0]$$

Metric

- The unknown in Einstein equations is g_{ij} – called *metric*. It describes a “distance” (called *interval*) in space-time between two infinitesimally close events:

$$ds^2 = \sum_{i=0}^{3}\sum_{j=0}^{3} g_{ij}dx^i dx^j \equiv g_{ij}dx^i dx^j$$

- In flat Minkowski space-time:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

- Metric is a *tensor*: a matrix-valued function.

Dynamical Space

- In GR space becomes a dynamical quantity. Space can be curved, perturbed, deformed in arbitrary way, and these deformations can change with time.
- Distortions of space can move – those are called ***gravitational waves***.
- Space can reconnect with itself – ***wormholes***.
- Space can flow into a point of infinite density – ***singularity*** – making a ***black hole***.

Testing GR

- Any physical theory must be constantly tested, and GR is no exception.
- All tests of GR can be separated into two types: a *weak field limit*, i.e. testing GR when deviations from Newton’s gravity are weak, and a *strong field limit*, when deviations from the Newton’s law are large.
- Weak field limit test are numerous (but they are less valuable, because there are alternative theories of gravity).

Weak Field Tests

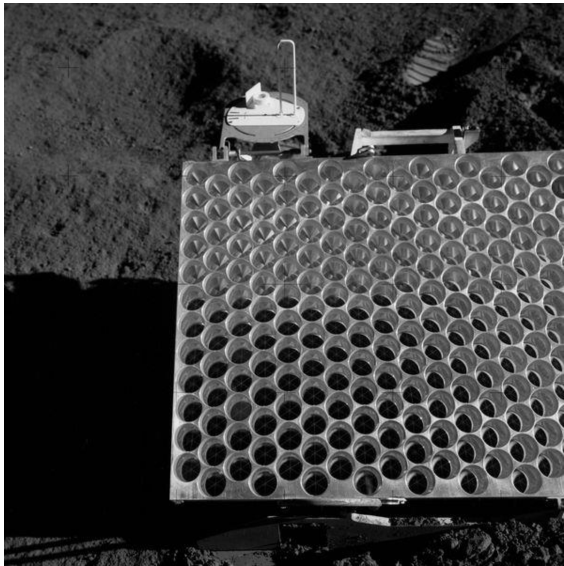
- Precession of planetary orbits.
  - Anomalous precession of Mercury explained by Einstein in 1916.
- Bending of light by the Sun’s gravity field.
  - Measured by Eddington in 1919.

Weak Field Tests

- Time delay due to the Earth gravity.
  - Need to account in time-keeping arose in ~1960.
  - Delay due to Saturn measured by *Cassini* probe to 0.002% in 2003.

Weak Field Tests

- Gravitational redshift.
  – Pound–Rebka experiment at Harvard in 1959.
  – Routine correction for modern GPS systems.
- “Frame dragging”.
  – Gravity Probe B (2005) – to 15%
  – LARES satellite (2012) – to 1%.
- Lunar Laser Ranging Experiment
  – Measures everything.

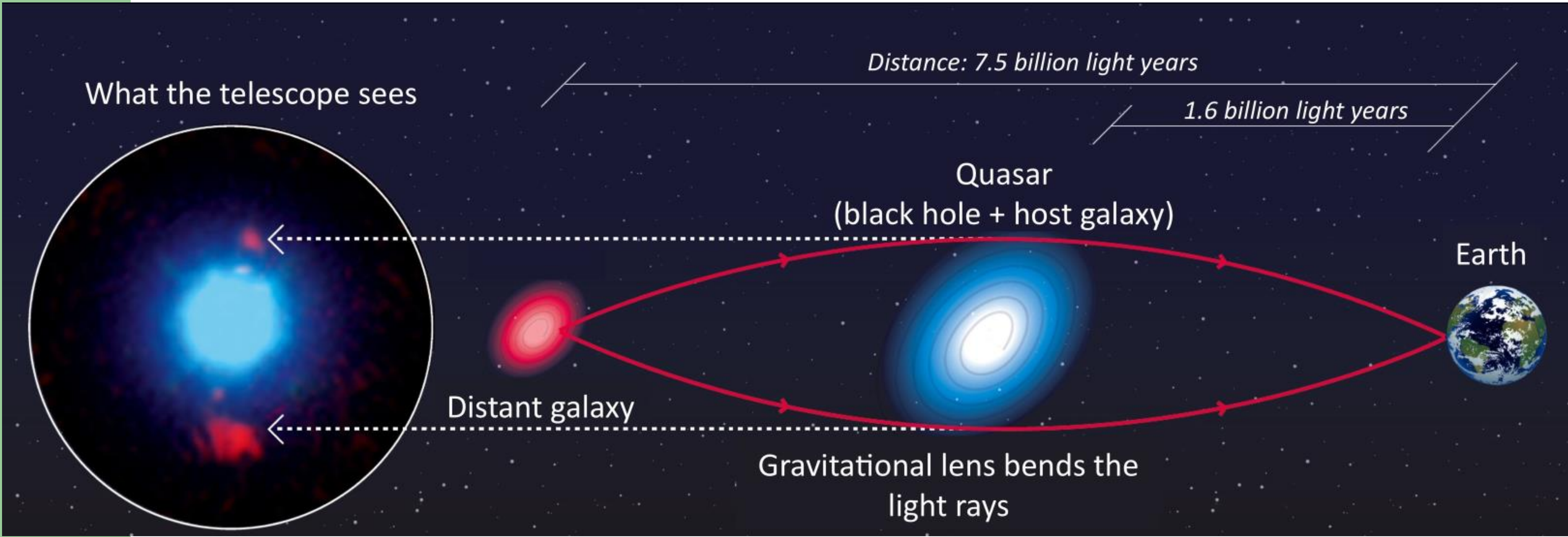


Strong Field Tests

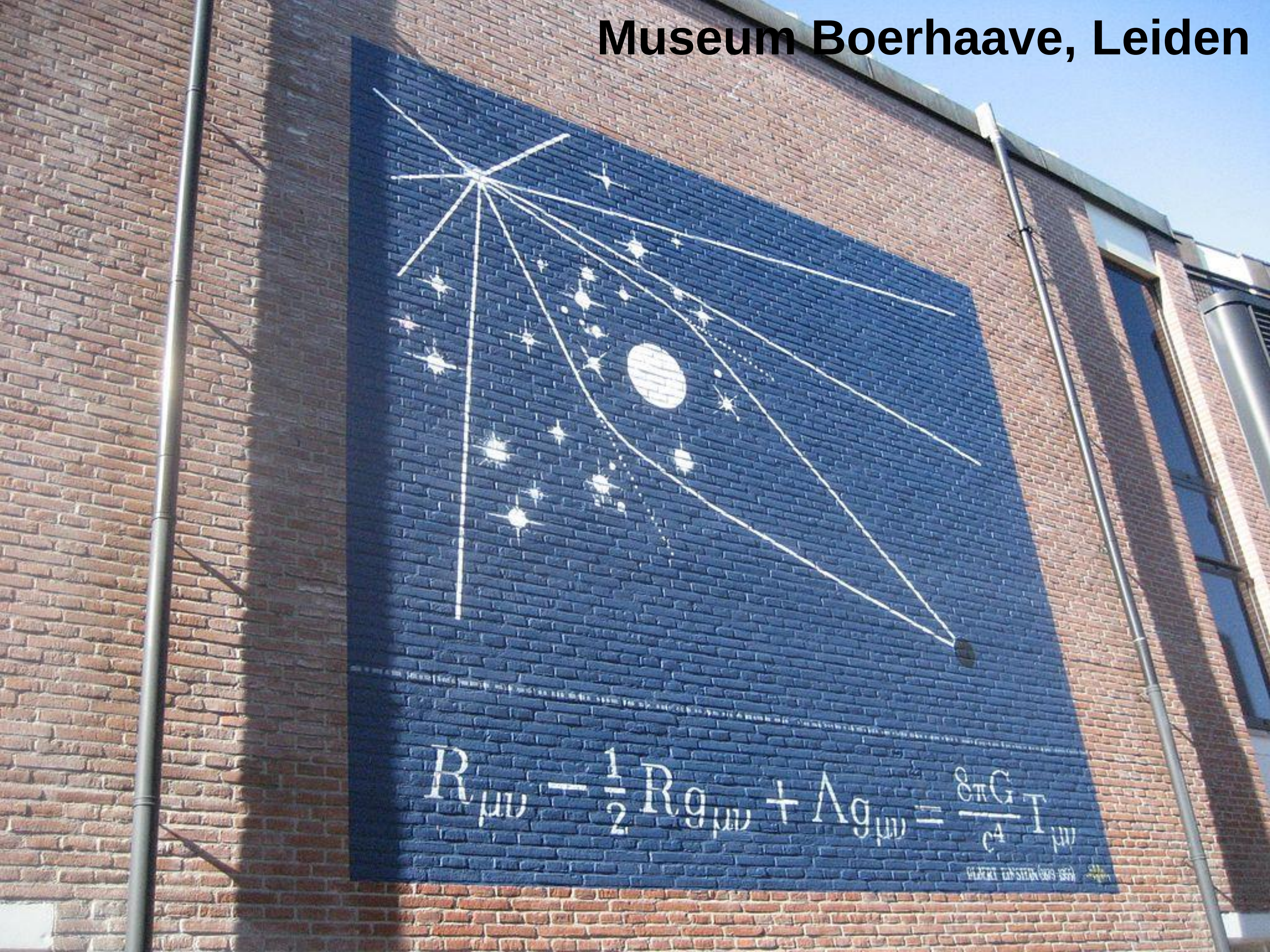
- Strong field limit tests are much more difficult to perform, but only they can convincingly confirm (or reject!) GR:
  - Gravitational radiation from a binary pulsar.
    - First discovered in 1974 by Joseph Taylor and Russel Hulse (Nobel Prize in 1993) - TBD.
  - Existence of black holes.

Gravitational Lensing

- Since mass bends light, and since galaxies are really massive, there must exist astronomical examples of light bending – “gravitational lensing”.

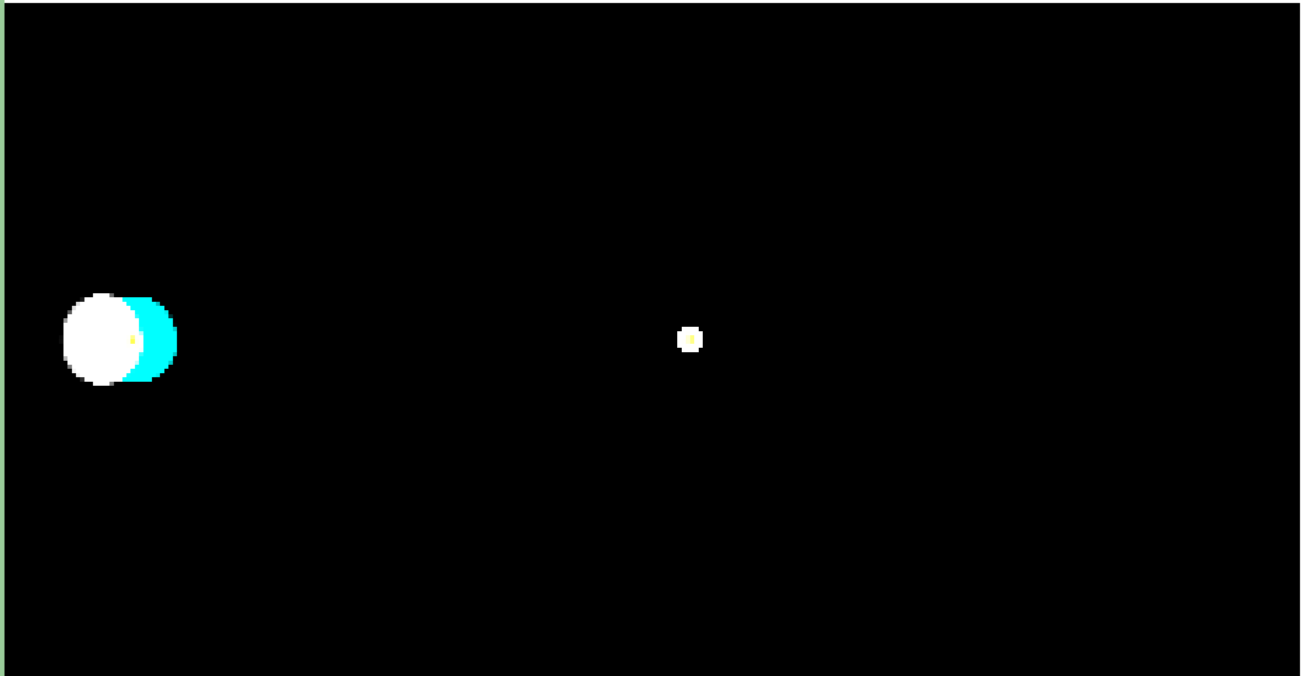


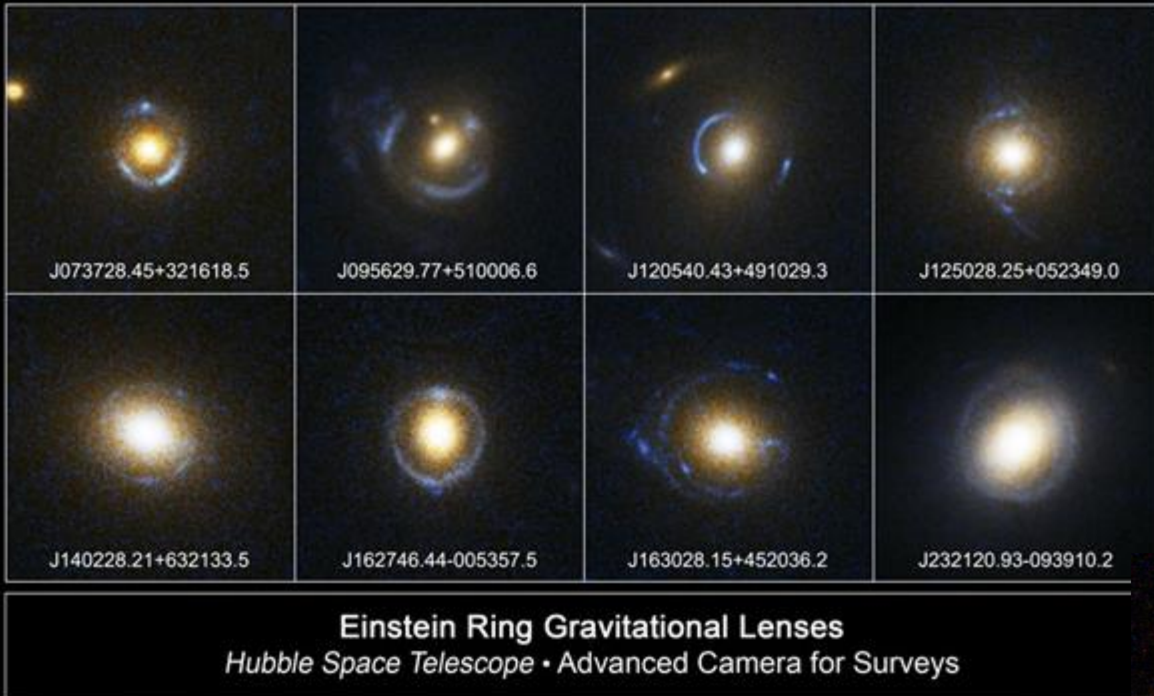
Museum Boerhaave, Leiden

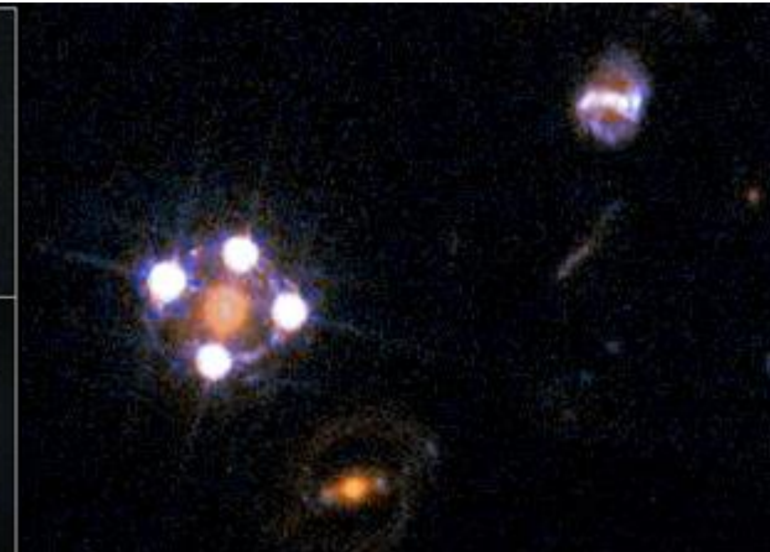


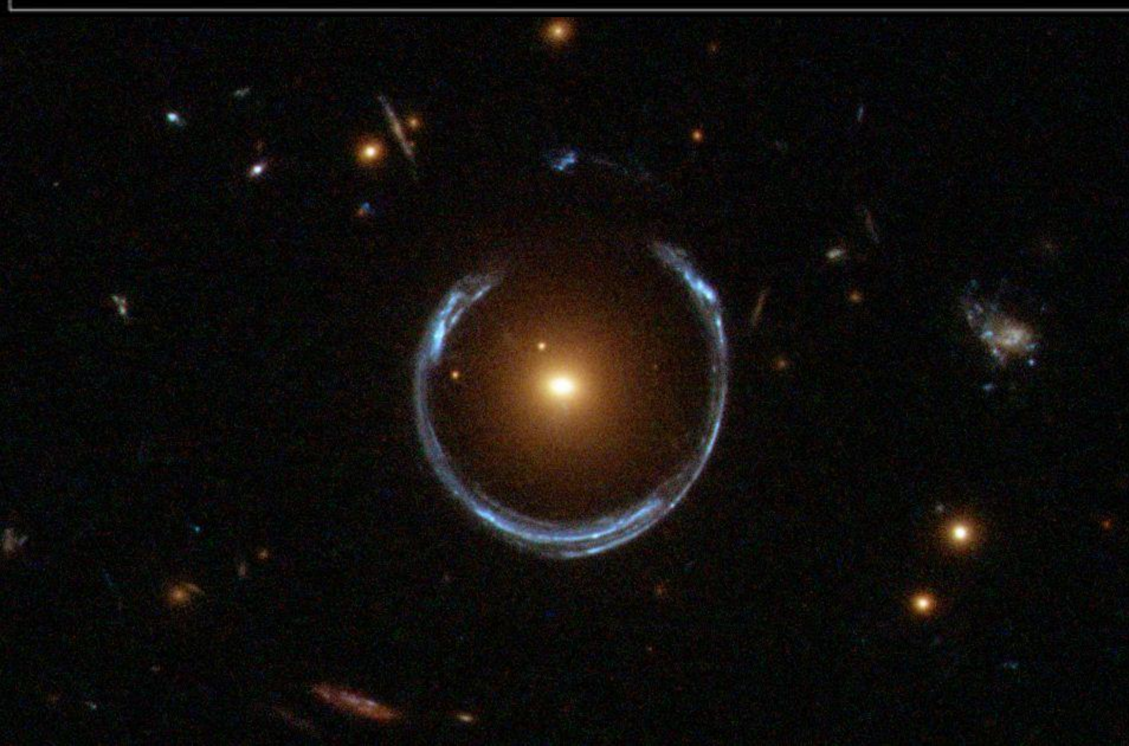
Gravitational Lensing

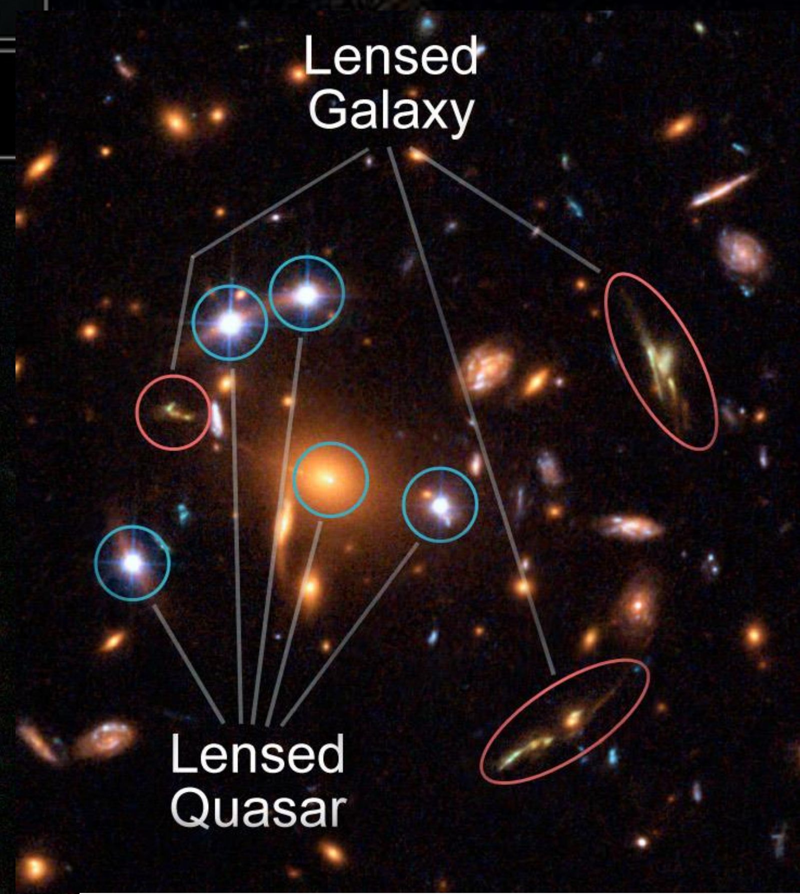
- Lensing results in various distortions to the "source image" – multiple images, arcs, or "Einstein rings".




J073728.45+321618.5
J095629.77+510006.6
J120540.43+491029.3
J125028.25+052349.0
J140228.21+632133.5
J162746.44-005357.5
J163028.15+452036.2
J232120.93-093910.2
Einstein Ring Gravitational Lenses
Hubble Space Telescope • Advanced Camera for Surveys






Lensed Galaxy
Lensed Quasar

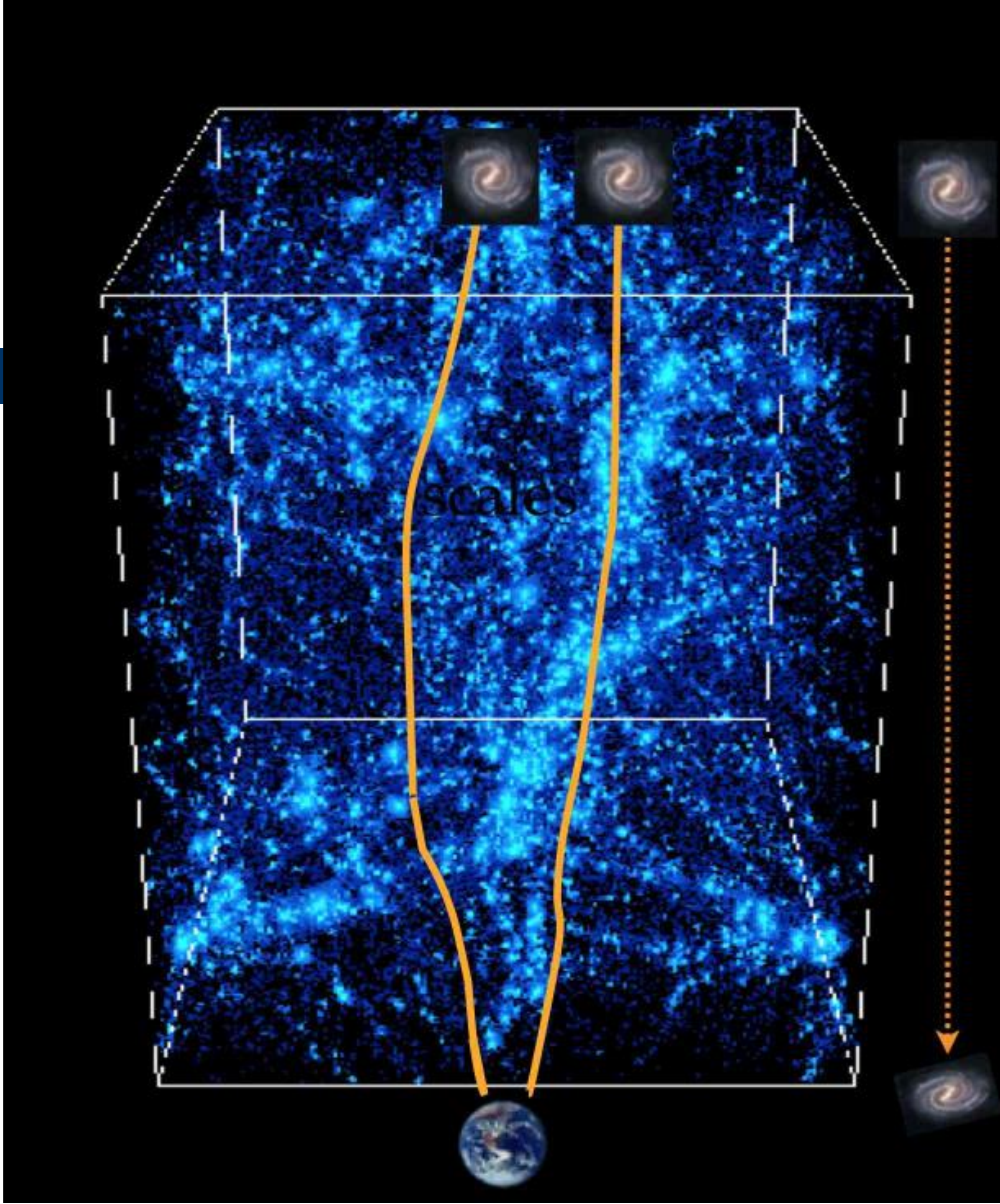
Celestial Smiley Face

Gravitational Lensing

- Gravitational lensing is a powerful tool for astronomers to
  - look for the faintest galaxies;
  - map the mass distribution of the universe (dark or not, here I observe!);
  - measure masses of galaxies and clusters of galaxies;
  - measure the expansion rate of the universe.

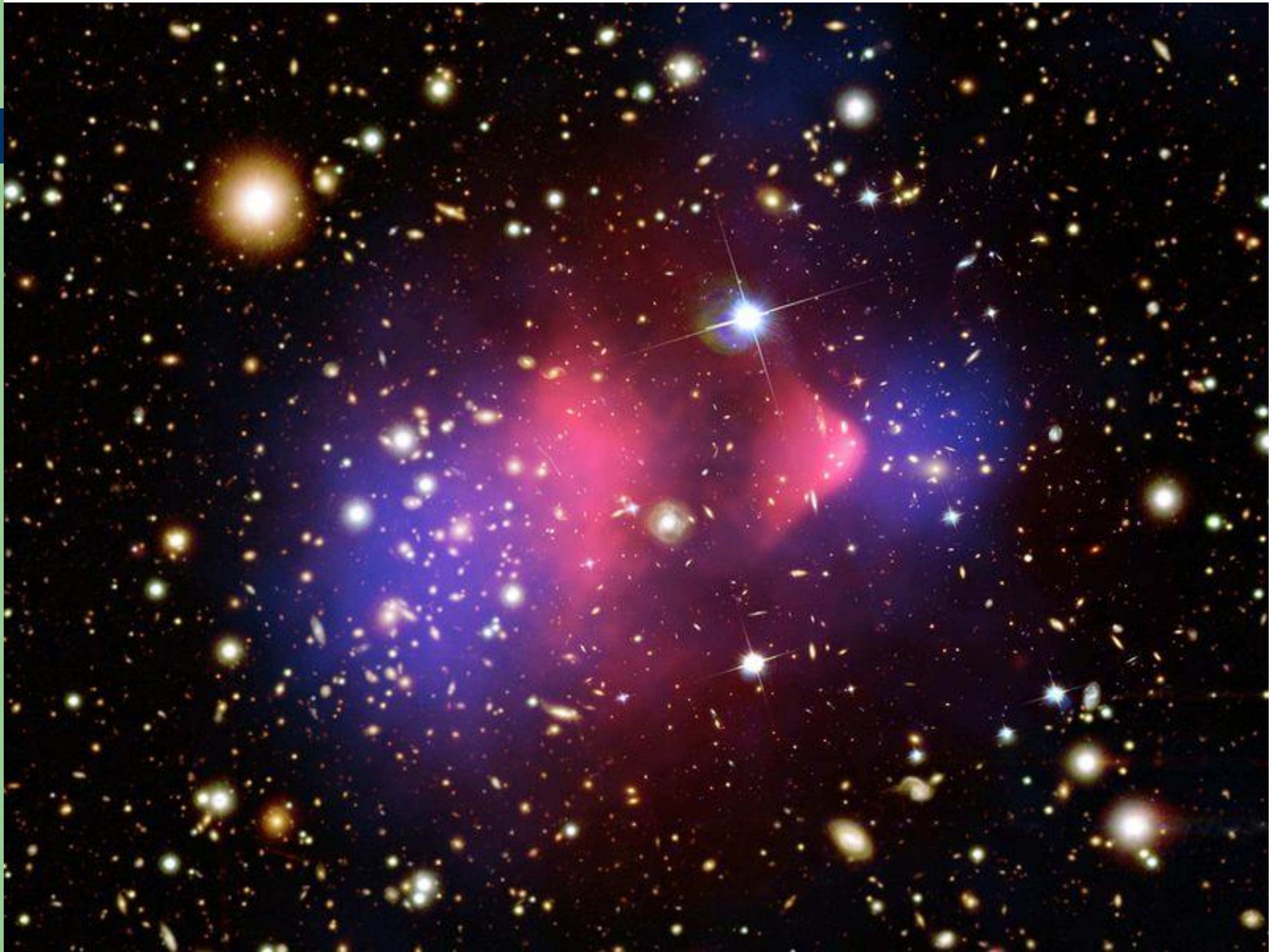
Weak Lensing

- Tiny changes in shapes of millions of galaxies.



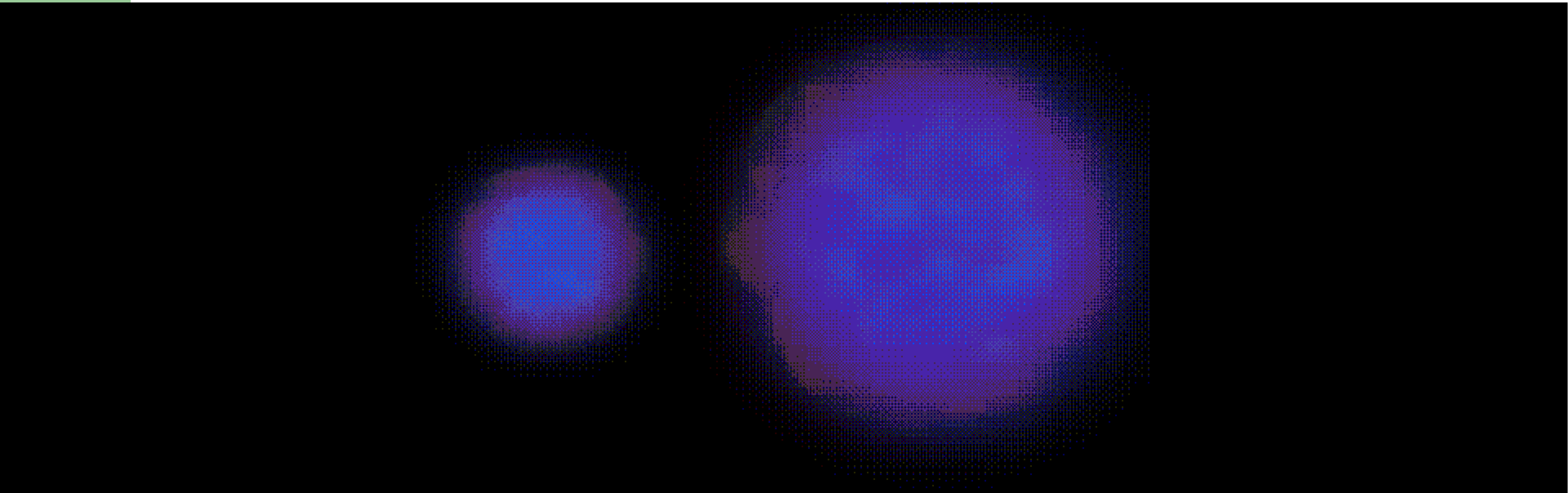
Weak Gravitational Lensing

“Bullet Cluster”



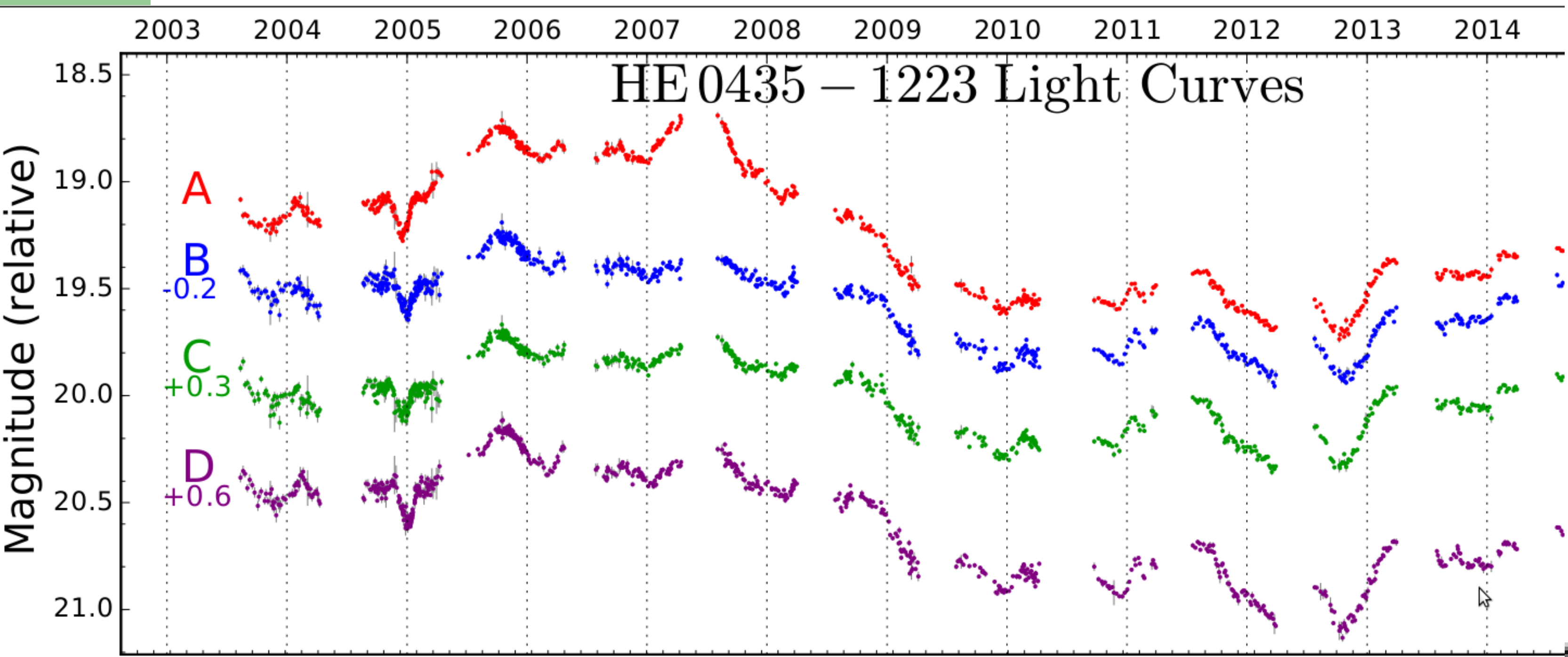
“Bullet Cluster”

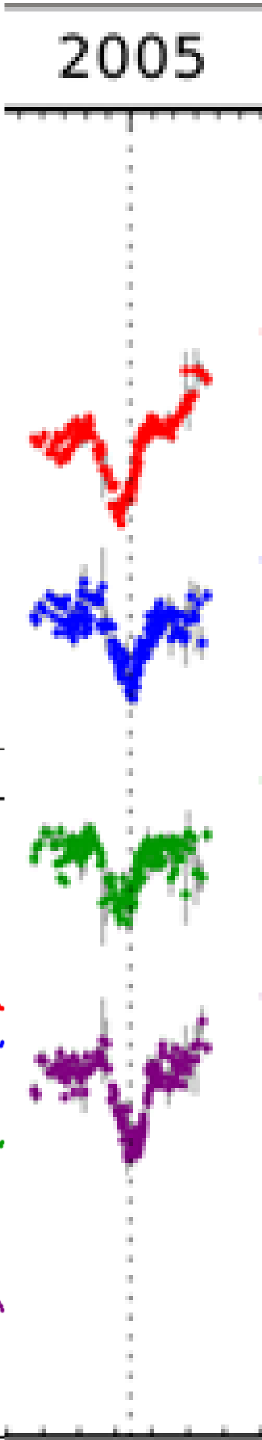
- One direct evidence for the existence of “dark matter” – matter that does not interact with baryons too much.



Gravitational Lensing

- Can be used to directly measure distances to quasars.



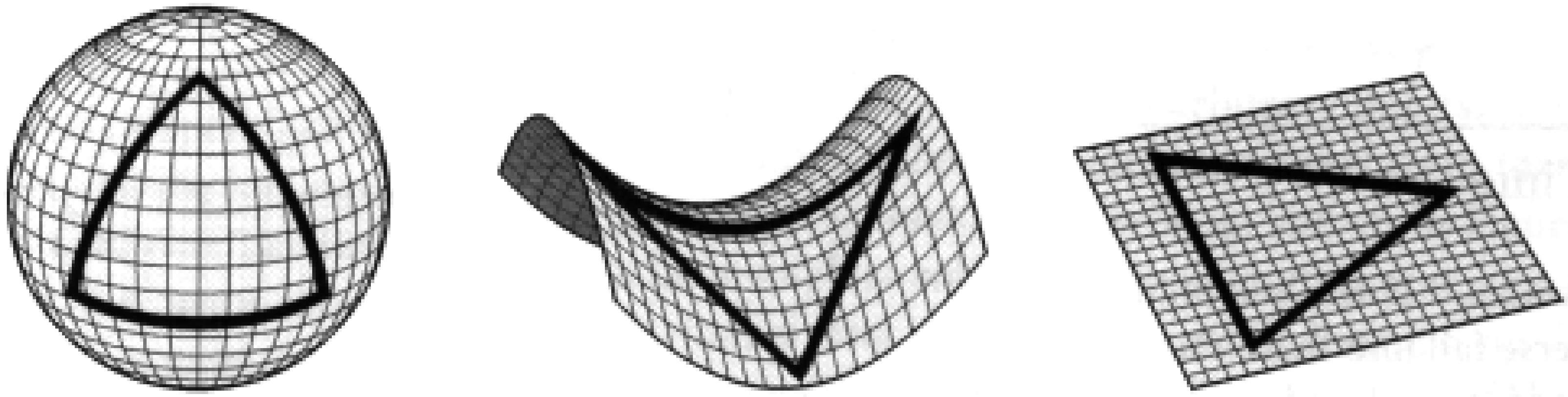


Cosmology

- Cosmology is one area of astronomy where it is all about General Relativity.
- We know that on large scales the distribution of matter in the universe is uniform (*homogeneous and isotropic*).
- That implies that the curvature of space may be positive, negative, or zero, but it is the same everywhere.

Cosmology

- Positive curvature – “closed universe”
- Negative curvature – “open universe”
- Zero curvature – “flat universe”



Positive Curvature   Negative Curvature   Flat Curvature

Cosmology

- We know from astronomical observations that our universe is **flat** to within the error of the measurement (about 0.07%). What does this imply for the time evolution of the universe?
  A. Universe must be expanding
  B. Universe must be contracting
  C. Universe must be static
  D. There is no connection between spatial curvature and time evolution of the universe.

Cosmology

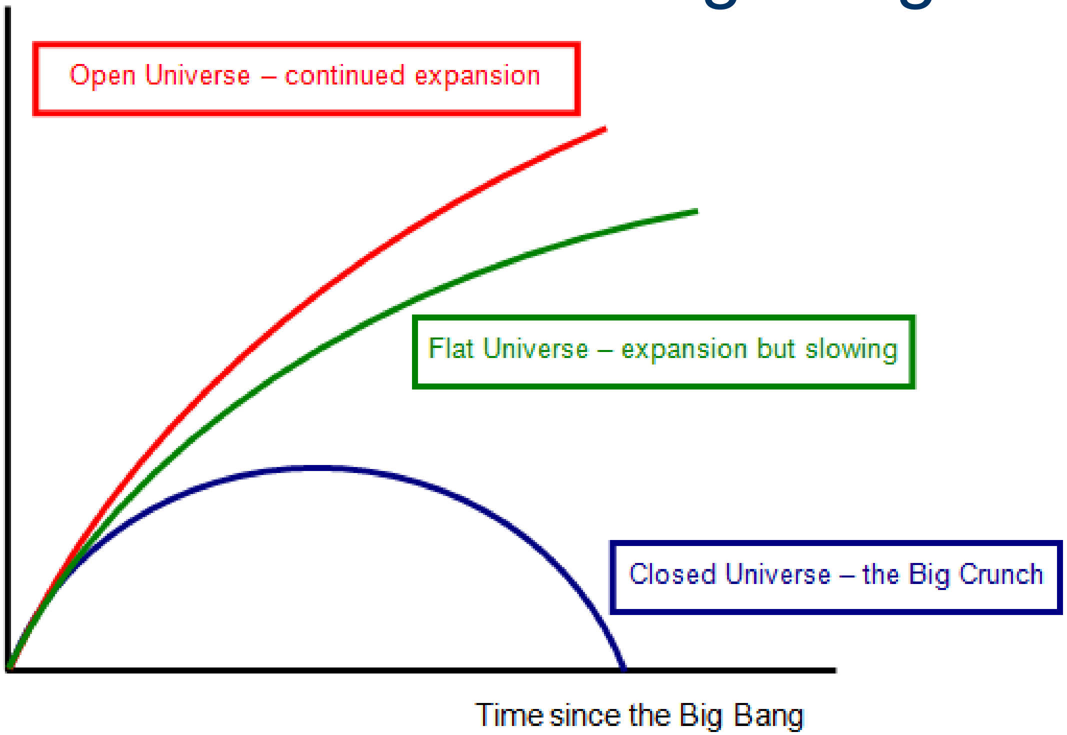
- Recall Einstein equations:

Curvature = Matter + Energy

- Since the universe contains matter and energy, space-time must be curved. If the space is flat, the time part of space-time must be curved – i.e. the universe cannot be static, it must expand or contract.
- This is true even if the universe is not flat.

Cosmology

- Evolution of distance between any two inertial observers – hence the “Big Bang”.

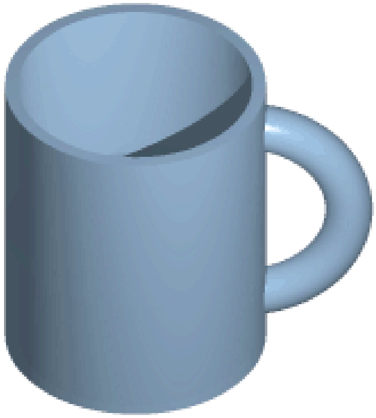


Cosmology

- The full story is way more complicated, with dark matter, dark energy, inflation, and other weird stuff.
- We will not venture there; we have our own weird stuff to take care of…

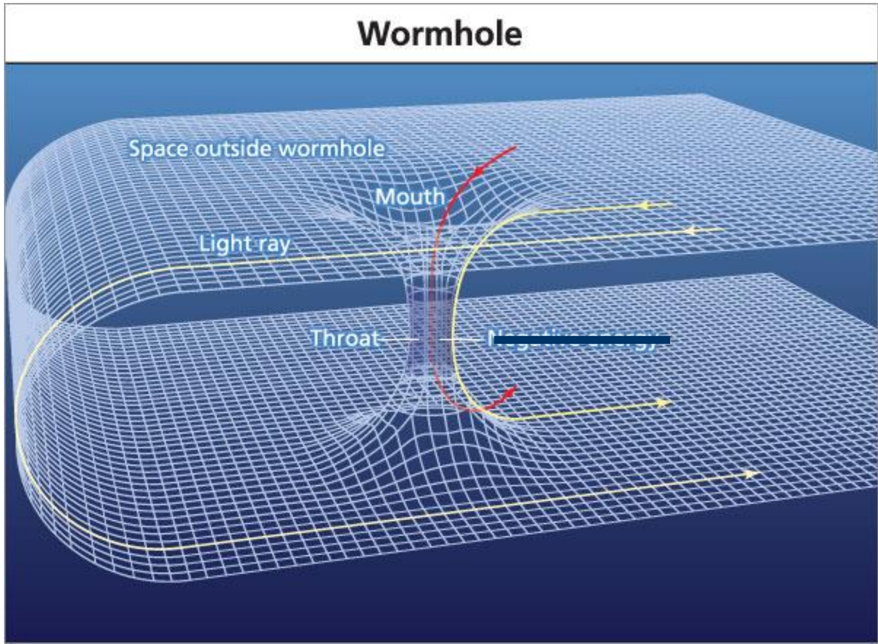
Time Travel and Other Crazy Things

- There is a part of mathematics called “topology”. It studies the most general properties of geometric shapes that are preserved in continuous transformations (a torus is **not** a sphere).



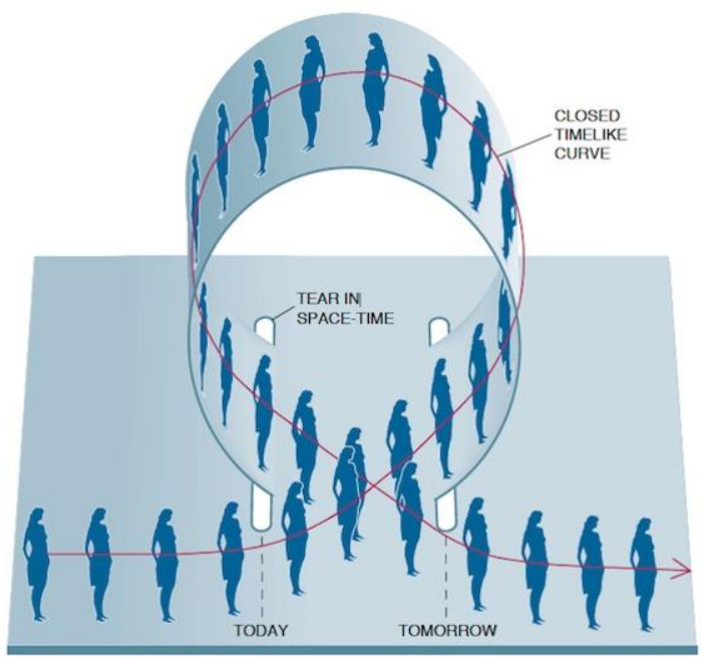
Time Travel and Other Crazy Things

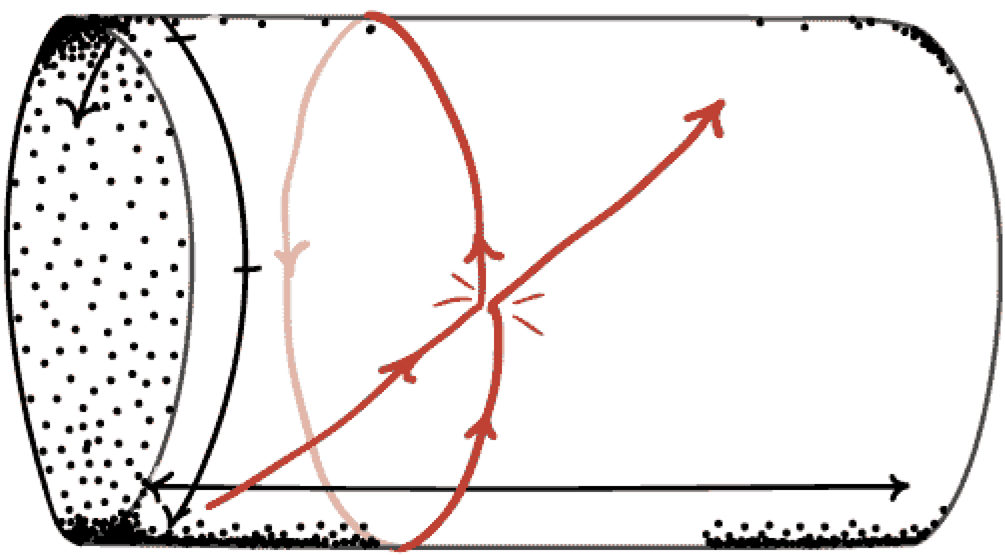
- Einstein equations determine the metric of space-time, but they do not determine its topology – the topology of space-time is not part of GR.
- One example – a wormhole.
- Another example – a closed time-like world line.



Time Travel and Other Crazy Things

- Einstein equations do not forbid time-like loops. You cannot kill your father, though, as there is just one space-time.





c.